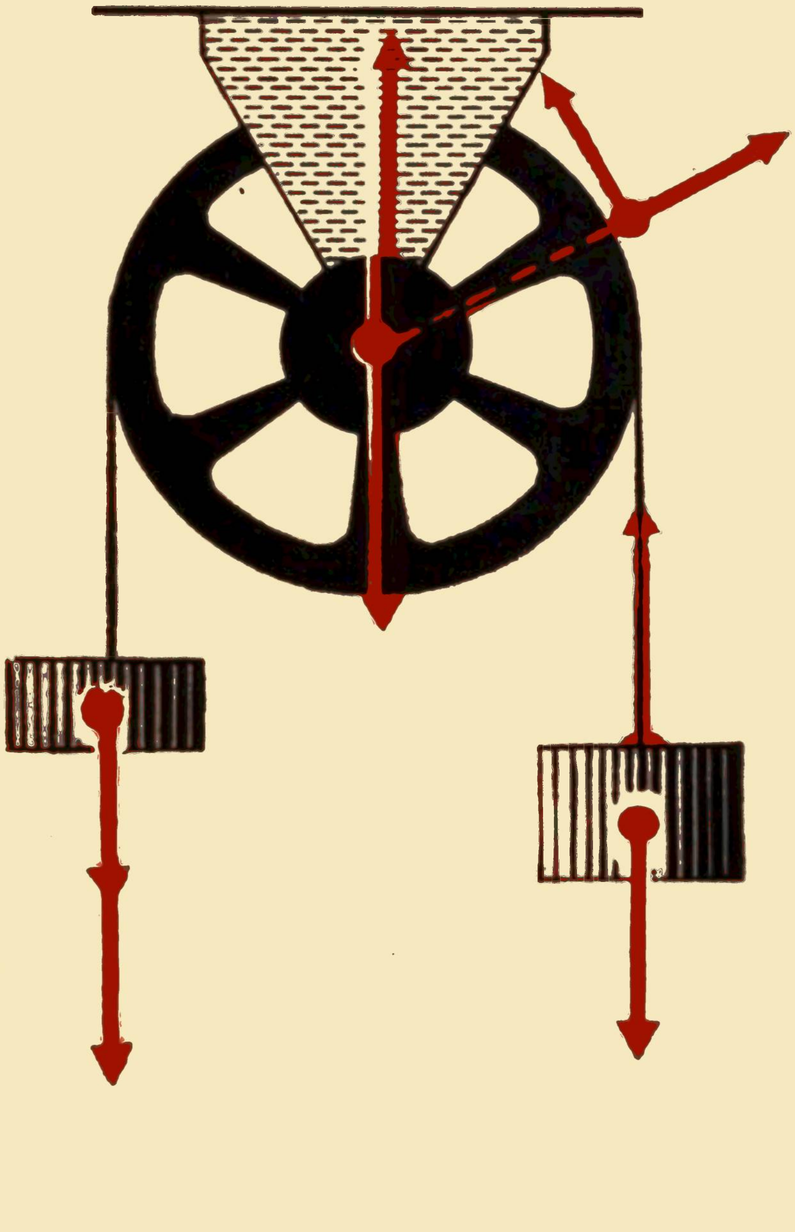
THEORETICAL MECHANICS

E. M. Nikitin



MIR PUBLISHERS · MOSCOW

The text of *Theoretical Mechanics* follows closely the course of study of theoretical mechanics in mechanical engineering technical schools (intermediate level).

The basic concepts and principles of theoretical mechanics are thoroughly explained and their relationships with engineering practice and the world about us are dealt with at length. Detailed solutions are given to a large number of typical problems, and numerous hints and suggestions are supplied throughout.

This textbook is designed for both full-time and correspondence students of technical schools, but is also suitable as a self-instruction course in the fundamentals of theoretical mechanics.

THEORETICAL MECHANICS

E. M. Nikitin

Е. М. НИКИТИН

ТЕОРЕТИЧЕСКАЯ МЕХАНИКА

(ДЛЯ ТЕХНИКУМОВ)

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

E. M. NIKITIN

THEORETICAL MECHANICS

for technical schools

Translated from the Russian
by George Yankovsky

MIR PUBLISHERS • MOSCOW

First published 1980
Revised from the 1977 Russian edition

На английском языке

© Главная редакция физико-математической литературы
издательства «Наука», 1977

© English translation, Mir Publishers, 1980

CONTENTS

PREFACE 11

INTRODUCTION 13

Sec. 1. The subject and method of theoretical mechanics . . 13
Sec.*2. A brief historical survey 17
Sec. 3. The basic laws of classical mechanics 20
Sec. 4. The concept of force 24
Sec. 5. Scalar and vector quantities 26

Part I

STATICS

CHAPTER I INTRODUCTION TO STATICS 30

Sec. 6. The subject and the axioms of statics 30
Sec. 7. Constraints and reactions of constraints 35

CHAPTER II PLANE SYSTEM OF CONVERGING FORCES 43

Sec. 8. Converging forces. The composition of two forces applied at a single point 43
Sec. 9. Decomposing a force into two converging components 46
Sec. 10. Polygon of forces 51
Sec. 11. The projection of a vector on an axis. Determining a vector from its projection 55
Sec. 12. The projection of a geometric sum of vectors on an axis 59

Sec. 13. An analytic determination of the resultant of a plane system of converging forces 60
Sec. 14. Conditions of equilibrium of a plane system of converging forces 63
Sec. 15. A theorem on the equilibrium of three nonparallel forces lying in one plane 64
Sec. 16. Remarks on problem solving that involves the equilibrium of a system of forces 65
Sec.*17. Truss analysis. Essentials. The method of isolating joints 72

CHAPTER III SYSTEMS OF TWO PARALLEL FORCES . 78

Sec. 18. The composition of two parallel forces acting in the same direction 78
Sec. 19. The composition of two forces unequal in magnitude and acting in opposite directions 80
Sec. 20. Resolving a force into two parallel component forces 83

CHAPTER IV THE THEORY OF COUPLES IN A PLANE. THE MOMENT OF A FORCE RELATIVE TO A POINT . 86

Sec. 21. Couples 86
Sec. 22. The moment of a force about a point (centre) 89
Sec. 23. Properties of couples 91
Sec. 24. The composition of couples. The condition of equilibrium of couples 94

CHAPTER V FORCES ARBITRARILY LOCATED IN A PLANE 98

Sec. 25. Poinsot's theorem on the parallel translation of a force 98
Sec. 26. Reducing a plane system of forces to a single centre. The principal vector and the principal moment . . 99
Sec. 27. Reducing a plane system of forces to a single couple . 102
Sec. 28. Reducing a plane system of forces to a resultant . . 104
Sec. 29. The conditions of equilibrium of an arbitrary plane system of forces. Varignon's theorem on the moment of a resultant. The condition of equilibrium of a lever 108
Sec. 30. Different forms of equations of equilibrium of an arbitrary plane system of forces 112
Sec. 31. Remarks on solving problems involving the equilibrium of a plane system of forces 114

Sec. 32. The equations of equilibrium of a plane system of parallel forces 120
Sec. 33. The equilibrium of a system of linked bodies . . . 123
Sec.*34. Determining forces in the bars of a truss by the method of cuts. 127
Sec.*35. Statically determinate and statically indeterminate problems 131

CHAPTER VI FRICTION 135

Sec. 36. Two basic types of friction 135
Sec. 37. Sliding friction 136
Sec. 38. The angle and cone of friction 143
Sec. 39. Rolling friction 146

CHAPTER VII A SPATIAL SYSTEM OF FORCES 152

Sec. 40. A spatial system of converging forces 152
Sec. 41. The moment of a force about an axis 161
Sec. 42. The conditions of equilibrium of a system of forces located at random in space 165
Sec. 43. The equations of equilibrium of a spatial system of parallel forces 170
Sec. 44. The equilibrium of a body having a fixed axis . . . 173

CHAPTER VIII THE CENTRE OF PARALLEL FORCES AND THE CENTRE OF GRAVITY OF A BODY 178

Sec. 45. The centre of parallel forces 178
Sec. 46. The centre of gravity of a body 181
Sec. 47. The coordinates of the centre of gravity of a body. The static moment of the area of a plane figure . . 183
Sec. 48. The centre of gravity of a symmetric body 186
Sec. 49. The position of the centre of gravity of certain homogeneous bodies of simple shape 187
Sec. 50. Determining the position of the centre of gravity of figures and bodies of intricate shape 191

CHAPTER IX STABILITY OF EQUILIBRIUM 199

Sec. 51. The concept of stability of equilibrium of a body having a point of support or an axis of rotation . . . 199
Sec. 52. The stability of a body resting on a plane 201

Part II

KINEMATICS

CHAPTER X INTRODUCTION TO KINEMATICS 207

Sec. 53. The subject and essentials of kinematics 207
Sec. 54. Methods of specifying the motion of a point 210

CHAPTER XI THE VELOCITY OF A POINT 218

Sec. 55. The concept of the velocity of a point 218
Sec. 56. Determining the velocity of a point in the natural method of specifying motion 221
Sec.*57. Finding the velocity of a point from the equations of motion in rectangular coordinates 223

CHAPTER XII THE ACCELERATION OF A POINT . . 229

Sec. 58. The concept of acceleration of a point 229
Sec. 59. Determining the acceleration of a point whose motion is specified by the natural method. Tangential and normal accelerations 233
Sec.*60. Determining the acceleration of a point from the equations of its motion in rectangular coordinates . . . 241

CHAPTER XIII SPECIAL CASES OF THE MOTION OF A POINT 247

Sec. 61. Uniform motion of a point 247
Sec. 62. Rectilinear nonuniform motion of a point 248
Sec. 63. Uniformly variable motion of a point 256

CHAPTER XIV ELEMENTARY TYPES OF MOTION OF A RIGID BODY 267

Sec. 64. Translational motion 267
Sec. 65. Rotational motion 270
Sec. 66. The trajectories, velocities and accelerations of points of a rotating rigid body 276
Sec. 67. Special cases of rotational motion of a rigid body . 281
Sec. 68. Transmissions of rotational motion 286

CHAPTER XV COMPOUND MOTION OF A POINT . . . 299

Sec. 69. Absolute, relative and migratory motion of a point . 299
Sec. 70. Theorem of composition of velocities 302
Sec. 71. Resolving the velocity of a point into components . 307

CHAPTER XVI COMPOUND MOTION OF A RIGID BODY 310

Sec. 72. The concept of compound motion of a body 310
Sec. 73. Plane-parallel motion of a body 310
Sec. 74. Resolving the motion of a plane figure into translational motion and rotational motion. The relationship between the velocities of distinct points of the figure 313
Sec.*75. A theorem on the projections of the velocities of two points of a figure 319
Sec. 76. The instantaneous centre of velocities of a figure . . 320
Sec. 77. Distribution of velocities of points of a plane figure . 324
Sec.*78. Composition of rotations about parallel axes . . . 329
Sec.*79. The angular velocity as a vector. The analogy between the composition of angular velocities and the composition of forces 337
Sec. 80. Planetary gear trains and differential gears 339

Part III

DYNAMICS

CHAPTER XVII INTRODUCTION TO DYNAMICS 344

Sec. 81. The subject of dynamics and its two basic problems 344
Sec. 82. The basic laws of dynamics 346
Sec. 83. Systems of units 353

CHAPTER XVIII ESSENTIALS OF KINETOSTATICS (DYNAMIC FORCE ANALYSIS) 357

Sec. 84. D'Alembert's principle 357
Sec. 85. The concept of an inertial force 359
Sec. 86. Forces of inertia in the curvilinear motion of a particle 360

CHAPTER XIX THE WORK AND POWER OF A FORCE . 369

Sec. 87. The work of a constant force over a rectilinear path . 369
Sec. 88. The work of a variable force over a curvilinear path. A graphical depiction of the work 372
Sec. 89. A theorem on the work of a resultant 376
Sec. 90. The work of the force of gravity 378
Sec.*91. The work of an elastic force 381
Sec. 92. Power. Efficiency 384
Sec. 93. The work and power of a force applied to a rigid body rotating about a fixed axis 387

CHAPTER XX GENERAL THEOREMS ON THE DYNAMICS OF A POINT 392

Sec. 94. Momentum and impulse of a force 392
Sec. 95. A theorem on the change of momentum of a particle . 393
Sec. 96. Potential and kinetic energy 398
Sec. 97. A theorem on the change in kinetic energy of a particle 400

CHAPTER XXI ELEMENTS OF THE DYNAMICS OF A SYSTEM 407

Sec. 98. External and internal forces of a system 407
Sec.*99. The law of conservation of momentum of a system 409
Sec.*100. The centre of mass of a system. A theorem on the motion of the centre of mass of a system 412
Sec. 101. The basic equation of dynamics for rotational motion of a rigid body 420
Sec. 102. The moment of inertia of a body 425
Sec.*103. A theorem on the moments of inertia of a body about parallel axes 428
Sec.*104. The moments of inertia of certain homogeneous bodies of elementary geometric shape 430
Sec. 105. A theorem on the change in kinetic energy of a system 433
Sec. 106. The kinetic energy of a rigid body 435

INDEX 443

PREFACE

In this text we have devoted considerable attention to a discussion of the basic concepts and propositions of theoretical mechanics and their relationship with reality and engineering practice. The numerous examples, remarks and footnotes to be found in the text are designed to help the student towards a better understanding of the subject and a firmer grasp of the material; they are also aimed at averting false impressions and distorted conceptions as the student masters the subject.

Much care has been taken to develop the student's skills and habits in applying the propositions of mechanics to the solution of specific questions and problems. This aim is served by general suggestions and hints of a methodological nature and also by the 118 typical problems provided with solutions and occupying about 25% of the book.

The textbook offers a full coverage of all aspects of the obligatory portion of the syllabus of theoretical mechanics for mechanical engineering technical schools. It also discusses a number of problems that go beyond the obligatory portion of the syllabus but are important in certain fields. The sections and items that are not obligatory (they constitute about 15% of the whole text) are marked with an asterisk. They may be omitted without any loss of continuity of the remaining material.

All definitions and important conclusions are given in italics and it is hoped that this will serve to fix such propositions more firmly in the mind of the student (after he has fully mastered the contents of the problem at hand, naturally).

A few words are in order on the system of units used in this textbook. The International System of Units (abbreviated SI) is employed throughout. However, since the engineering system of units is still in use, the author has found it expedient to provide basic information about that system as well. As concerns derived units, they are expressed both in the SI and in the engineering system in appropriate sections on dynamics. A relationship is established between the two systems for the purpose of conversion from one system to the other.

INTRODUCTION

Sec. 1. The subject and method of theoretical mechanics

Motion is the mode of existence of matter, its basic inherent property.

By motion in its broad sense we understand not only the displacement of bodies in space but also thermal, chemical, electromagnetic and any other changes and processes, including our consciousness and thought.

Mechanics is the study of the simplest and most readily observable form of motion, mechanical motion.

Mechanical motion is defined as any change, in time, of material bodies relative to one another, and also any change relative to the position of the particles of one and the same material body (its deformation).

Of course, it would be wrong to reduce the great multiplicity of natural phenomena to mechanical motion alone and to explain them on the basis of pure mechanics. Mechanical motion does not by any means exhaust the essence of various forms of motion, but it is always present in every form and must be the first to undergo investigation.

Due to tremendous advances in science and engineering it has become impossible to concentrate in one subject the study of the vast multitude of problems that involve the mechanical motion of diverse material bodies. Today, mechanics is a whole complex of general and special engineering disciplines devoted to the study of the motion of separate bodies and systems of bodies, to the design and analysis of diverse structures, mechanisms, machines, and so forth.

The material bodies involved in these disciplines are extremely diversified, but their motions possess many properties that are independent of the physical properties of the moving bodies themselves. For instance, one can speak of the velocities of points of a body quite apart from any knowledge of the body itself, whether it be a particle of a fluid or gas, or a piston, a bullet or a spaceship. Again, one can speak of the rotation of a rigid body without going into the question of whether it is a wheel or a planet.

It is these general properties of mechanical motion of material bodies that are studied in theoretical mechanics.

Theoretical mechanics is the science that studies the general laws of mechanical motion of material bodies and establishes general procedures and methods for solving problems involving such motion.

In order to establish the laws of motion common to all material bodies, theoretical mechanics resorts to the device of generalization; this amounts to isolating that which is of primary importance and rejecting all subsidiary circumstances that are of no importance under the conditions at hand. For this reason, in theoretical mechanics we do not consider the motion of the physical bodies that actually exist in nature, but of certain abstract models that reflect only specific common properties of the actual physical bodies. One such model is the material point (particle), another is the absolutely rigid body; these are precisely what are studied in the general course of theoretical mechanics.

All material bodies occupy a definite portion of space, which means they have specific dimensions. Generally speaking, the separate parts of the bodies may not execute the same motions. For example, the points of a pulley at different distances from its axis move in circles of different radii and with different speeds. But the smaller the size of the body, the less the motions of its different parts will differ. In the abstract, we can imagine a material body of infinitesimal proportions.

A material body whose size can be disregarded under the conditions of a given problem is termed a material point (or particle). In theoretical mechanics, a particle is not only some minute particle of a body, but at times can be a body of appreciable size. The main thing is that the dimensions of the body do not play a significant part in the investigation at hand. For instance, when studying the motion of the earth about the sun, we can regard the earth as a particle, and, since the earth is small compared with its distance from the sun, we can regard all particles of the earth as covering the same distances in that motion. As we shall see later on, any body in translational motion (that is, motion in which all its particles move in the same manner) can be regarded as a material point (particle).

All real bodies undergo deformation under the action of external forces, and so to ensure rigidity and reliability of structures and machines in operation, the material and dimensions of the parts are chosen so that deformations under load are extremely small. In many problems, such insignificant deformations may be ignored and one can regard the distances between the particles of the given body as constant. This brings us to the concept of an absolutely rigid body.

An absolutely rigid body is one in which the distance between any two points remains constant.

In nature, there is no such thing as a material point or an absolutely rigid body. These are merely abstractions that of course do not reflect in toto the properties of specific physical bodies. But one need not attempt to consider all properties, as long as those that are neglected do not appreciably affect the nature of the motion under study.

If at every step we attempted to take into account all the properties of a concrete physical body, the problem would become so complicated that it would be impossible to solve. But if we first establish, through theoretical mechanics, the general laws of motion based on a consideration of the motions of a particle and of absolutely rigid bodies, we can then apply them to concrete physical bodies.

It is only by abstracting oneself from the particular, the individual, that which is peculiar to single objects and phenomena that one is able, in making observations, to establish general regularities, in particular, the general laws of mechanics. This is done by generalizing the results obtained. To illustrate, Galileo started with separate observations of the motion of falling bodies and generalized the results to obtain a general law of motion for all bodies falling in a vacuum.

This, as V. I. Lenin pointed out, is the general path of development of any true science: "From living perception to abstract thought, *and from this to practice*,—such is the dialectic path of the cognition of *truth*, of the cognition of objective reality." (V. I. Lenin, *Collected Works*, Vol. 38, p. 171).

The simplified scheme that we use in theoretical mechanics to replace an actual physical body does not depend only on the properties of that body but also on the type of problem that we are investigating. For instance, when studying the earth's motion about the sun, we can, as mentioned above, assume the earth to be a particle, but that cannot be done if we are engaged in a study of the earth rotating on its axis. This is because the particles of the earth are at different distances from the axis and cover different distances in their motion.

It is customary to divide the course of theoretical mechanics into three basic parts: statics, kinematics and dynamics. *Statics is the study of the rules of composition of forces and the conditions of equilibrium of rigid bodies. Kinematics is the study of motions of bodies from the geometric viewpoint alone, without reference to the forces acting on the bodies. Dynamics is the study of the relationships between the motions of material bodies and the forces acting on them.*

Theoretical mechanics not only makes it possible to explain a number of important phenomena in the surrounding world, but also serves as a scientific foundation for many engineering disciplines. Its methods and procedures are employed in engineering calculations used to design all manner of structures and machines and their operation. The role and significance of theoretical mechanics in engineering in general is constantly growing. The extremely complicated engineering problems that constantly arise in modern technological development, and the organization and expansion of radically new types of production and technical equipment cannot be solved on the basis of experimental findings and practice alone. What is required is scientific foresight and rigorous preliminary calculations based on a profound knowledge of theory, primarily a knowledge of the laws and methods of theoretical mechanics. Thus, besides the important general educational value of theoretical mechanics, we find that it plays a tremendous role in the development of the technician as a specialist. The better and more profoundly the student masters the basic propositions of theoretical mechanics, the more meaningfully and freely will he employ its methods and the easier it will be for him to take up the study of specialized engineering subjects.

Sec. *2. A brief historical survey

Together with astronomy and mathematics, mechanics is one of the oldest sciences. The gigantic pyramids of Egypt that were erected several thousand years before the Christian era and the remnants of still more ancient structures in China, India and other countries all stand witness to the fact that in ancient times use was made of such mechanical contrivances as levers, rollers, pulleys and other devices needed to simplify the moving of heavy bodies. True, at that stage the general laws of mechanics could not have been discovered. Mechanics as a science came into being with the first writings that theoretically summarized accumulated human experience.

As social production advanced and changed and as technology developed, the science of mechanics underwent significant change in content as well.

The term "mechanics" (from the Greek *mēchanē* meaning expedient, machine) was first introduced by the great Greek philosopher of antiquity Aristotle (384-322 B. C.). However, though his writings on motion and forces contained certain correct propositions, there were also many false ones, and on the whole these works were not of a scientific nature.

It is to Archimedes (287-212 B. C.), the greatest mathematician and mechanician of ancient Greece, that we turn in seeking the founder of mechanics, mainly statics, as a precise science. Archimedes made a number of outstanding discoveries in mathematics and mechanics. For one thing, he gave an exact solution to the problem of the lever and established the doctrine of the centre of gravity. Archimedes discovered the law bearing his name that deals with the pressure exerted by a liquid on a body immersed in it.

The methods he developed for measuring the surface areas and volumes of bodies of various shapes evolved into integral calculus two thousand years later.

Archimedes combined remarkable theoretical discoveries with marvellous inventions, some of which are still in use today.

In the Middle Ages the development of mechanics, like that of other natural sciences, came to a halt due to the stifling forms of feudal production.

Mechanics flourished in the Renaissance between the end of the 15th century and the beginning of the 16th century, during the great upsurge in trade. The development of trade brought with it the construction of roads, the building of ships, the expansion of industry, the development of navigation and military affairs. All this served as a stimulus for the rapid development of science as a whole and mechanics as an important part of it.

The heliocentric system of the world that was created by the great Polish astronomer Nicolaus Copernicus (1473-1543) revolutionized scientific thinking. According to the Copernican view, the sun was placed at the centre of the universe, and the earth and other planets revolved about the sun and also rotated on their respective axes. This represented a fundamental break with the religious views of the earth as the centre of the universe and served as a basis for the development of the new science of celestial mechanics and the development of dynamics as the most important branch of theoretical mechanics.

Up to that time, all the discoveries in mechanics had involved studying the laws of equilibrium, or statics.

The beginnings of dynamics are linked closely with the name of the great Italian Galileo Galilei (1564-1642), a passionate adherent of the teachings of Copernicus. Galileo was the first to demonstrate that under the action of a constant force, a body moves with uniformly accelerated motion and not, as thought since the time of Aristotle, uniform motion. He also stated the law of inertia. Galileo experimentally established the law of bodies falling in a vacuum; he solved the problem of the motion of a body thrown upwards at an angle to the horizon; and others. The investigations that Galileo made into the relationship between the dimensions of structural elements and the loads that they can sustain served as the starting point of a new science called strength of materials.

The great English scientist Isaac Newton (1643-1727) completed the development of the ideas of Galileo and his successors and established the basic laws of classical mechanics (which is theoretical mechanics based on the three main laws of Galileo and Newton). Newton introduced the concept of mass and gave a precise statement of the second law, which is the basis of all dynamics. To Newton again goes the credit for discovering two highly important laws of mechanics: the law of equality of action and counteraction (reaction) and the law of universal gravitation.

Besides establishing the basic laws of mechanics, Newton solved a large number of concrete problems in astronomy and mechanics. Newton was one of the creators of the differential and integral calculus, which exerted a tremendous influence on all subsequent development of mechanics. The appearance of differential and integral calculus (mathematical analysis) contributed greatly to the advance of mechanics in the 18th century.

An outstanding role in the development of analytical methods in mechanics was played by the works of the prominent French scientists Jean Le Rond d'Alembert (1717-1783) and J. L. Lagrange (1736-1813).

Outstanding contributions to mechanics have been made by Russian and Soviet scientists.

The great Swiss mathematician and mechanician Leonhard Euler (1707-1783), who spent many years of his life in Russia and was a member of the Russian Academy of Sciences, wrote over 800 scientific papers on mathematics, astronomy, rigid-body dynamics, hydromechanics and the strength of materials.

The prominent Russian mathematician and mechanician Academician P. L. Chebyshev (1821-1894) devised, among other things, new methods in the synthesis of mechanisms and laid the foundation of the famous Russian school of the theory of mechanisms and machines.

The great Russian scientist N. E. Zhukovsky (1847-1921) laid the foundations of aerodynamics. One of the outstanding representatives of Zhukovsky's marvelous Russian school of hydro- and aeromechanics was S. A. Chaplygin (1869-1942), who elaborated a series of problems in aerodynamics that are of exceptional importance to modern high-speed aviation. Zhukovsky and Chaplygin also made highly significant contributions to other fields of mechanics as well.

I. V. Meshchersky (1859-1935) is well known for his pioneering work in creating a new and important field of theoretical mechanics, the mechanics of bodies of variable mass, which finds its main applications in rocketry.

K. E. Tsiolkovsky (1857-1935), a remarkable Russian scientist, is the founder of the modern theory of rocket flight (rocket dynamics) and the theory of interplanetary communications.

At the present time, advances are being made in mechanics by scientists around the world. A brilliant example of the attainments of Soviet scientists in various spheres of mechanics is the success of the USSR in the exploration of outer space.

Sec. 3. The basic laws of classical mechanics

Classical mechanics is the mechanics based on the three basic laws of Galileo and Newton. These laws are a generalization of conclusions drawn from numerous experiments and age-long observations of the motions of material bodies.

Practice itself corroborates the conclusions drawn from these laws. Structures built on the basis of the basic laws of classical mechanics stand strong, machines operate, devices function properly, ocean vessels ply the waters of the world, and aircraft fly. This serves as irrefutable proof of the correctness of the propositions and laws of mechanics that were used in the calculations, proof of the fact that the laws of mechanics do indeed reflect objective processes of nature that are independent of man's consciousness and, hence, are themselves objective laws.

However, the laws of mechanics are not to be regarded as some sort of immutable, "eternal" truths. By virtue of the imperfections of human experience, they are only an approximation to absolute truth. The principle of dialectical materialism that every real science contains truths that only approximate absolute knowledge as humanity progresses fully applies, of course, to mechanics as well. The advances made in physics at the beginning of this century demonstrated that the laws of classical mechanics do not possess the universality earlier attributed to them and cannot be applied to the motion of micro-particles or to the motion of bodies moving at velocities close to that of light. This led to a new mechanics based on the theory of **relativity***.

However, for the purposes of ordinary life, classical mechanics (a study of the fundamentals of which forms the content of this course) fully retains its value.

For the time being, we will confine ourselves to a consideration of two basic laws of mechanics (the first and third laws) and only insofar as they are needed to grasp the concept of a force and for the study of statics. In the last part of this course (dynamics) we will expand the investigation of these laws and will go into a detailed examination of the second basic law of mechanics.

The first law (or law of inertia). *A particle*** *remains at rest or in a state of uniform motion in a straight line unless acted upon by an external force.*

The fact that a particle remains in a state of uniform rectilinear motion in the absence of any other particles acting on it would seem to contradict everyday experience and was discovered only in the 17th century by Galileo. We observe that a body (particle) left to itself (or so it appears) gradually loses speed and finally comes to a halt. A closer look however shows quite clearly that this apparent contradiction to the law can be explained by the effect of other particles (bodies) on the motion of the given particle: friction generated by the surface along which the particle is moving, the resistance of the air, and so on. These are what slow down the speed of our particle. If measures are taken to reduce the influence of such factors (resistance to motion), the retardation will be smaller and the motion will be closer to uniform motion. From this we can conclude that if we were able to eliminate all resistance to the motion of the particle, its velocity would remain unchanged in direction (hence, it would be rectilinear) and in magnitude (hence, it would be uniform).

---

* The theory of relativity, which was created by the great physicist A. Einstein (1879-1955), does not reject classical mechanics but rather includes it as a special case that holds true for macro-bodies moving with velocities appreciably less than that of light.

** The word "particle" here is meant to include any material body.

In order to subject a particle to such conditions, it is necessary to isolate it, which means removing all other particles from that region so that they cannot affect the motion of the given particle.

The interaction of particles which gives rise to a change in velocity of the particles or their deformation is said to be mechanical interaction.*

The measure of mechanical action on a particle by other particles that characterizes the magnitude and direction of the action is said to be a force applied to that particle.

In a practical sense, it is of course impossible to completely isolate a particle from all surrounding particles, that is, from the action of forces. There is no way in which we can completely eliminate the forces of resistance to the motion of a particle. For this reason, in order to maintain the motion of a particle it is necessary to apply a force acting in a direction opposite to the force of resistance. If this force is less than the force of resistance, the particle slows down and comes to a halt. If it is greater than the force of resistance, the particle picks up speed. And if the magnitude of the force is equal to that of the force of resistance (and is in the opposite direction), then the velocity of the particle remains unchanged. In that case we say that the applied forces are in equilibrium. It is clear that such forces do not in any way affect the motion of the particle and, hence, they may be disregarded when determining the motion of the particle.

From the above we can draw a conclusion that is important in practical application: *every particle under the action of forces in equilibrium maintains its velocity unchanged (in particular, it remains in a state of rest).*

Such a particle (or absolutely rigid body—one that does not undergo deformation) may be regarded as a particle to which no forces at all are applied.

---

* Only in magnitude, only in direction, or in magnitude and in direction. In all these cases we say that the velocity undergoes a change.

The converse also holds true: *If a particle is in uniform motion, that means all forces acting on it are in equilibrium.* Contrariwise, if a change of velocity of the particle takes place, then the forces acting on that particle are not in a state of equilibrium.

The *inertia of a particle* (or rigid body) *is the capacity of the particle to retain its state of motion* (magnitude and direction of its velocity or, in a particular instance, the state of rest) *in the absence of forces acting on it* (or if the forces are in equilibrium). Hence, the law that establishes that property of any particle is termed the law of inertia.

In antiquity, rest was regarded as a natural state of matter and it was assumed that any body set into motion from a state of rest would tend to return to that state. The Latin word "inertia" means lack of skill, idle. Today, of course, the word means something quite different in mechanics. The meaning of "inertia of a body" is simple and obvious: the motion of a body cannot by itself change without the action of forces. If even the slightest force* is applied to any body, the velocity of the body will change and will continue to change until the force ceases to act.

The third law (the law of action and reaction). *When two particles act on each other, their forces are equal in magnitude and act in opposite directions along a straight line.* In other words, to every action there is an equal and opposite reaction.

All particles in nature are interrelated and, hence, act on one another.

According to the third law, the actions of bodies (particles) are never one-sided. Thus, if a body is attracted to the earth, then the body itself attracts the earth with that same force. If a body experiences the resistance of a medium, that body acts on the medium with the same force and produces a displacement of the particles of the medium. In the friction generated between a body and the plane along which it is moving, both contacting surfaces experience the same rubbing action. If we regard one of the forces as the action, then the other force is the reaction.

---

* A force applied to a body is to be understood as the "excess" of a moving force over the forces of resistance.

The central idea to be mastered here is that there are no one-sided actions of forces in nature; there is only an interaction of bodies (particles). All forces operate in pairs. For this reason, though we make frequent use of such an expression as "applying a force to a body", it is to be understood to mean that some other body is acting with a given force on the body under consideration. And in the process, the given body invariably acts with an equal but opposite force. While focussing our interest on the motion of one body, we merely disregard for the moment its reaction on other bodies that serve as the source of forces applied to the given body (or particle).

Although the *forces of interaction of two bodies* are equal and opposite, they *do not balance* for the simple reason that they are not applied to a single body but to two distinct bodies.

If action and reaction were balanced, then it would never be possible to press on a body and set it in motion.

The equality of action and reaction mentioned in the third law should not be confused with the equality of their results. The result of the action of a force on a body is determined not only by the magnitude of the force but by a set of other circumstances as well: the mass of the body, its elastic properties, the presence of other forces acting on the given body, and so forth. Take the collision of an ocean liner and a small sailboat; their actions on each other will be the same, but the result of such actions will quite naturally be different.

Sec. 4. The concept of force

The concept of force arises out of everyday practice. At every turn, one perceives muscular effort needed to move an object from one place to another, to pick it up, to alter its speed of motion, and so forth. By analogy with this sensation of effort, we call force the measure of the action of one body on another such that the body changes its mechanical state.

We encounter a variety of forces in the world about us: the force of gravity, the attractive and repulsive forces of electrified and magnetized bodies, the force of friction, the force of pressure of one body on another, and so on. The question of why the different forces arise is the subject matter of different sections of physics. Theoretical mechanics is only interested in the mechanical effect, that is, the changes (produced by forces) in the mechanical state of a given body.

If the change in the mechanical state of a body amounts to a change in velocity, then we have what is sometimes spoken of as a dynamic manifestation of force.

For instance, the force of gravity acting on a body thrown upwards gradually reduces the velocity of the body and, contrariwise, a falling body increases its speed. Now, if a body is thrown at an angle to the horizon, the force of gravity acts on it and changes its velocity in magnitude and direction (a body in a vacuum moves in a parabola).

If the change in the mechanical state of a body is not a change in velocity but is a deformation, then we speak of a static manifestation of the force.

For instance, the force of gravity does not change the motion of a body if it is at rest on a horizontal table. The body interacts with the table as well, and the force of gravity acting on the body pulls it downwards and is balanced by the force of resistance of the table.*

|Experience teaches us that the *action of a force on a body is determined* by three elements: (1) the *point of application of the force*, (2) the *direction of the force*, and (3) the *magnitude* (or *numerical value*) *of the force.*

The point of application of a force is that material particle of a body on which the force acts.

Note that the notion of a point of application of a force is merely conditional since it is practically impossible to apply a force at a single geometrical (dimensionless) point. The forces dealt with in mechanics and regarded as being concentrated in a single point are actually the resultant of several forces acting on all points of a given part of a surface or a given volume of a body.*

---

* Note that here the balance is between the force of gravity (weight) of the body and the force of resistance of the table applied to the same body, and not between the force of pressure of the body on the table (action) and the force of resistance of the table (reaction) applied to different bodies.

The direction of a force is the direction of that rectilinear motion that the force would impart to the point of its application if that particle of the body were free and were originally in a state of rest.

For example, the force of gravity is directed vertically downwards because that is the direction in which bodies at rest fall in the absence of other forces acting on them.

The straight line along which the force is directed is termed the line of action of the force.

The magnitude of a force is found by comparing it with some other force that is taken as unity.

Devices used in static measurement and comparison of forces are called dynamometers. Spring scales are a simple dynamometer. The operating principle here is that (to a certain extent) the deformation (tension) of a spring is proportional to the acting force and vanishes when the force is removed.

In the International System of Units (abbreviated SI, for Système International d'Unités), which is now obligatory in the USSR, the *unit of force is the newton*, N.

To convert from the still frequently used kilogram-force units (abbreviated kgf) to newtons, note that 1 kgf $\approx$ $\approx$ 9.81 N. A good thing to remember is that a force of 1 N is roughly equal to one tenth the force of 1 kgf.**

The choice of international unit of force is based on the so-called dynamic definition of force, which is why the justification for that choice will be given in Sec. 83 of Dynamics.

Sec. 5. Scalar and vector quantities

Two substantially different types of physical quantities are distinguished.

One type (examples are temperature, time, mass, density, area, volume) is characterized numerically (for a chosen unit of measure). For instance, to describe the temperature of the air at a given time, it suffices to measure it in, say, degrees Celsius. The resulting number (positive or negative) is the value of the temperature.

---

* For example, the force of gravity acting on a body is understood to mean the resultant of the forces of gravity on all its particles.

** 1 kgf is the weight taken in the metric system for the international prototype of mass at the point on the earth where the acceleration of a freely falling body is $g = 9.80665 \approx 9.81$ m/s^2.

Quantities that are fully described by a single numerical value are termed scalar quantities, or *scalars.*

The other type of quantity requires more than magnitude to give it a sufficient description: namely, direction as well.

Quantities defined not only by magnitude but also by direction in space are termed vector quantities, or *vectors.*

Vectors include not only force but a number of other mechanical quantities such as velocity, acceleration.

A vector quantity is graphically depicted as a straight line (*AB* in Fig. 1), the length of which is the magnitude (to a chosen scale) of the vector and the direction of which coincides with the direction of the vector. In drawings, direction is indicated by an arrowhead.

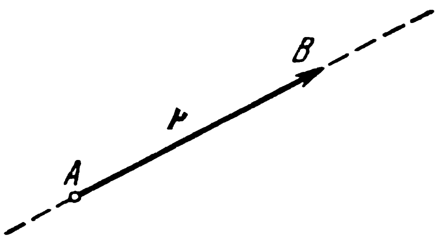


FIG. 1

The numerical value of a vector is termed the modulus or *absolute value* or *magnitude.* The magnitude of a vector is a positive scalar quantity.

Points A and B of the line segment AB (Fig. 1) *depicting a vector are called the origin and the terminus of the vector AB*, respectively.

Since operations involving vectors are essentially different from operations on scalar quantities, vector quantities will be distinguished by boldface letters; for example, **P**. Sometimes, a vector will be denoted by two lightface letters topped by an arrow; for example, $\overrightarrow{AB}$. Here, the first letter denotes the origin of the vector and the second letter the terminus.

The magnitude of a vector will be designated by the same letter as the vector itself, but lightface and the two letters will not be topped by an arrow. For example, P will be used to denote the magnitude of the vector **P**, and the letters AB will denote the magnitude of the vector $\overrightarrow{AB}$.

Three different types of vector quantities (vectors) are distinguished:

1. Localized (or bound) vectors are those whose origins are bound to a definite point of space, that is, the vectors have a definite point of localization.

2. Sliding vectors are those whose origins may be placed at any point on a specified straight line having the direction of the given vector (its line of action). Thus, sliding vectors may be translated along their lines of action.

3. Free vectors are those which can have any point in space as the origin. Such vectors are not bound to any definite line of action.

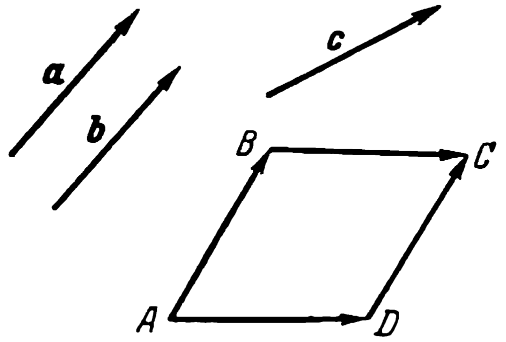


FIG. 2

Two vectors are said to be equal if they have the same magnitude and direction (that is, if they are parallel and in the same direction). For instance, vectors **a** and **b** (Fig. 2) are equal vectors, whereas **a** and **c** are not equal, though they have the same magnitude. This is because they have different directions.

In the rhombus $ABCD$ (Fig. 2), $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, but the vector $\overrightarrow{AB}$ is not equal to the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

Equal vectors are not ordinarily distinguished and are designated in the same manner.

It is important to point out that force cannot be regarded as a free vector because the point of application of a force cannot be chosen arbitrarily.

As in the case of vectors, forces may be subjected to a wide range of mathematical operations. The rules of such operations will be discussed at those points in the course where they are needed for an understanding of the propositions of mechanics under discussion (and only insofar as they are needed).

We conclude this section with a few words about the scale of a vector.

As already mentioned, a vector quantity may be graphically depicted as a straight-line segment of specific length and definite direction. The length of the segment must correspond to the numerical value (magnitude) of the vector to some scale (the nature and actual dimensions of the vector may be arbitrary). For this reason, an ordinary drawing scale is not suitable, generally speaking, for depicting vectors.

If some vector quantity Q is shown on a drawing by a line segment of length l, the scale μ of that quantity will be

$$\mu=\frac{Q}{l}.$$

From this formula it follows that the dimensionality of the scale of the vector is obtained by dividing the dimensions of the magnitude (Q) of the vector by the dimensions of the length l. The scale μ is ordinarily equipped with an index indicating the quantity it refers to. If, for example, it is required to depict graphically a force $P = 200$ N by a line segment of length $l = 10$ mm, then the scale of the force is $\mu_P = 20$ N/mm. For a velocity $v = 20$ m/s depicted by a line segment $l = 10$ mm, the scale will be $\mu_v = 2$ m/(s·mm) and so on.

PART I

STATICS

CHAPTER I

INTRODUCTION TO STATICS

Sec. 6. The subject and the axioms of statics

Statics is that part of theoretical mechanics which deals with the general properties of forces and the conditions of equilibrium of rigid bodies under the action of forces applied to them.

In statics, the equilibrium of a rigid body is to be understood as the state of rest of the body relative to a system of coordinates that is assumed to be fixed. Such a system in statics is usually the system of coordinates rigidly attached to the earth.

Fundamental to statics are, in addition to the first and third laws of classical mechanics, several so-called axioms of statics that have found confirmation over many centuries of practical applications. They serve as the basis for constructing all other propositions of statics by means of logic. Let us first agree on the following definitions.

1. *A collection of forces acting on a given body is called a system of forces. The individual forces of the system are called components of that system.*

2. *If the system of forces is such that a free body acted upon by those forces does not change its motion* or, as a special case, remains at rest, *then that system of forces is said to be a balanced system.* Speaking of the forces of such a system, we say that they are in equilibrium.

A "free" body (or particle) is one that is not bound to any other bodies, that is, one that may be moved at liberty in space.

3. *A force which, when joined to some system of forces acting on a body, brings the system to equilibrium is called a balancing force relative to the given system of forces.* Quite obviously, in a balanced system, each of the forces is a balancing force with respect to all other forces.

4. *Two systems of forces are termed equivalent if they exert the same mechanical action on one and the same free rigid body.*

5. *A single force that is equivalent to a given system of forces is said to be the resultant of that system.*

6. *Forces acting on a given body via other bodies are termed external forces. The forces of interaction between the particles of a given body are termed internal forces.*

First axiom. *An absolutely rigid body is in equilibrium under the action of two forces if and only if the forces are equal in magnitude and opposite in direction along a single straight line.*

One must bear in mind that this axiom, like all statements of statics, is unconditionally applicable only to an absolutely rigid body. When applied to actual deformable bodies, it is necessary to take into account the specific features of the forces and the bodies to which they are applied.

If, for example, we apply two equal and opposite forces $\mathbf{P}_1$ and $\mathbf{P}_2$ to the ends of a flexible thread, it will remain in equilibrium only when the forces are applied as shown in Fig. 3*a* (when the thread is under tension). If the forces are applied as shown in Fig. 3*b*, the thread will crumple and will not be in a state of equilibrium. Now if we take a rigid rod instead of a thread, then it will be in equilibrium in both cases shown in Fig. 3.

Second axiom. *Any balanced system of forces may be added to a system of forces or removed from that system without altering the action of the given system of forces on an absolutely rigid body.*

In other words, by adjoining any balanced system of forces to a given system of forces acting on a rigid body, we obtain a system that is equivalent to the given system. Conversely, if there are several forces forming separately a balanced group and if that group is part of a given system of forces, then the group may be removed (or disregarded). The remaining system is equivalent to the given system.

First corollary. *Any force applied to an absolutely rigid body at some point may, without affecting the action it exerts, be translated to any other point lying on the line of action of that force.*

Proof. Suppose (see Fig. 4) a body is acted upon by a force **P** applied to a point A. On the line of action of the force, take some arbitrary point B and apply to it two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ that are equal in magnitude to the force **P** ($P = P_1 = P_2$) and opposite in direction along the straight line AB. The forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal in magnitude and opposite in direction. By the first axiom of statics they mutually balance; hence, by the second axiom, the state of the absolutely rigid body will not undergo any change if they are adjoined to the system. The forces **P** and $\mathbf{P}_2$ also mutually balance (on the basis of the first axiom) and so they may be removed without altering the state of the absolutely rigid body. There remains one force, $\mathbf{P}_1$, which is equal to the given force **P** and lies on the line of its action. Since the point of application of the force $\mathbf{P}_1$ (point B) was chosen arbitrarily on the line of action of the given force, the proof of the corollary is complete.

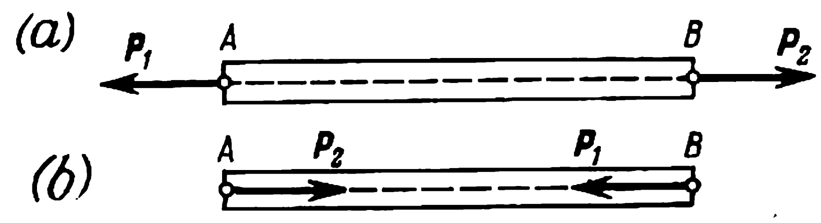


FIG. 3

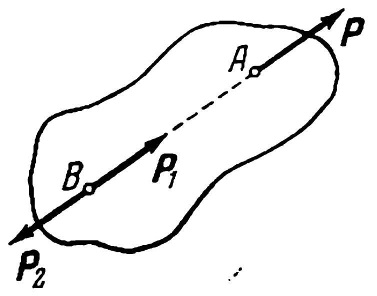


FIG. 4

We see that, for an absolutely rigid body, the point of application of a force is not essential and is replaced by the line of action of the force. Recalling what was said about types of vectors (see Sec. 5), we may note that a *force applied to an absolutely rigid body is a sliding vector.*

Thus, the action of a force on an absolutely rigid body is determined by the following elements: (1) magnitude, (2) line of action, (3) the direction of the force along the line of action. Of course, in each instance one can attribute to the force some point of application, but that point may be replaced by another point lying on the line of action of the force.

Note however that any adjoining and dropping of balanced forces and also any translation of the force along its line of action alter the pattern of distribution of internal forces in the body. For example, the change in the arrangement of forces applied to the ends of a rod as depicted in Fig. 3*a* to that depicted in Fig. 3*b* may be obtained by translating the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ along their lines of action to the respectively opposite ends of the rod. It is clear that for a deformable rod such a translation substantially alters the internal state of the rod since in the former case the rod is stretched, in the latter it is compressed.

As applied to actual physical bodies, the foregoing procedures may be used only when we consider the external action of forces on a given body, that is, when only the necessary conditions of equilibrium of the body are being determined. When considering these conditions we assume the body to be absolutely rigid and ignore the internal forces. Indeed, by the law of equality of action and reaction, any two particles of a body act on one another with forces that are equal in magnitude and opposite in direction along one straight line. Therefore, taken together these forces always constitute a balanced system, and in the case of an absolutely rigid body they may be ignored.

Since statics considers the equilibrium of absolutely rigid bodies and only such bodies, we will in future regard all forces acting on a body as external forces, unless otherwise stipulated.

Second corollary. *The resultant force and the balancing force are equal in magnitude and opposite in direction and act along a single straight line.*

Proof. Suppose that the force $\mathbf{R}_P$ is the resultant of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$ (Fig. 5). Let us apply along the line of action of the resultant force a force $\mathbf{R}'$ equal to it in magnitude and opposite in direction. Since the forces $\mathbf{R}_P$ and $\mathbf{R}'$ are equal in magnitude and opposite in direction, they balance. We can, without altering the state of the body, replace the resultant $\mathbf{R}_P$ by its components, that is, by an equivalent system of forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$. And since the force $\mathbf{R}'$ balances the resultant $\mathbf{R}_p$, it is a balancing force for the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ as well.

Since it is given that the force $\mathbf{R}'$ is equal in magnitude and opposite in direction to the force $\mathbf{R}_p$, the proof is complete. It follows from this corollary that finding the force which balances a given system of forces may be reduced to determining the resultant of that system.

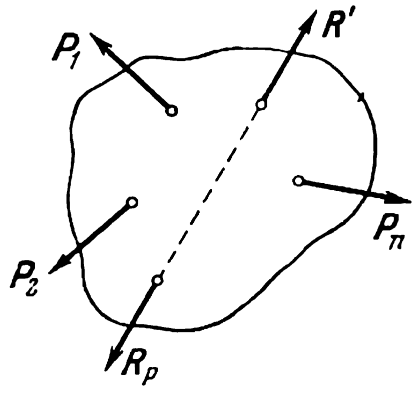


FIG. 5

Third axiom. *The resultant of two forces applied at one point is applied at the same point and is depicted by the diagonal of a parallelogram constructed on the given forces as sides.*

The parallelogram constructed on the given forces is termed a *parallelogram of forces*, and the rule for finding the resultant via construction of a parallelogram is called the *parallelogram rule.*

The composition of forces, like that of other vector quantities, by the parallelogram rule is called geometric (or vector) addition.

Geometric addition of forces, like the addition of any vectors, is denoted by the ordinary addition sign (+) between boldface letters (or letters with an arrow on top), which denote the forces.

If $\mathbf{R}_p$ is the resultant of two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied to a single point O of a body (Fig. 6), then on the basis of the given axiom we have

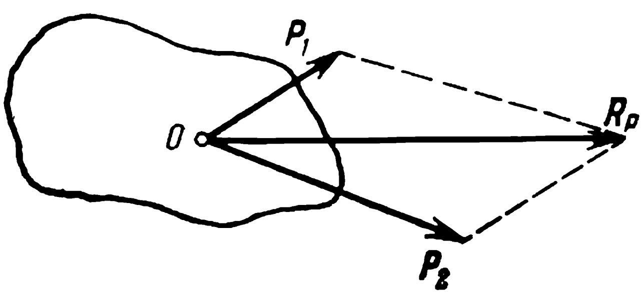


FIG. 6

$$\mathbf{R}_p = \mathbf{P}_1 + \mathbf{P}_2.$$

The third axiom of statics considers the resultant of two forces applied to a single point. If two forces are applied at different points of a body, but their lines of action intersect, then using corollary 1, we can translate both forces to the point of intersection of their lines of action and then add by the parallelogram rule. If the lines of action of the forces intersect somewhere outside the body, then, on translating both forces to their point of intersection and determining the resultant, we must transfer it along the line of action into one of the points of the body.*

Fourth axiom (the rigidity principle). *If a nonrigid body is in a state of equilibrium, that equilibrium will not be upset if the body becomes absolutely rigid.*

The axiom is obvious since any transformation of a nonrigid deformable body in equilibrium into an absolutely rigid body can only further restrict possible movements of the body and further fix the equilibrium of the body instead of upsetting it.

The rigidity principle permits applying to any nonrigid body and any variable structure the conditions of equilibrium established by statics for an absolutely rigid body. These conditions are the necessary conditions of equilibrium for nonrigid bodies as well, but they are not always sufficient conditions.

As mentioned above, for the equilibrium of a flexible thread it is not sufficient that the forces applied to its ends be equal in magnitude and opposite in direction along a straight line; it is further required that they stretch the thread and not compress it.

Thus, taking into account deformations that arise in an actual body under the action of forces applied to it merely supplements the results obtained in the mechanics of an absolutely rigid body but does not cancel them.

Sec. 7. Constraints and reactions of constraints

In most problems of mechanics, one deals with *bodies that are not free*, that is, with bodies that are in contact or are bound to other bodies, as a result of which certain motions of the given body become impossible.

If a body is not free, we say that constraints have been imposed on it. *Constraints are bodies that restrict the freedom of displacement of some given body.*

For a body lying on a table, the constraint is the table, for a shaft lying in bearings, the constraints are the bearings, for a ladder put up against a wall, the constraints are the wall and the floor of the room.

---

* This may not be possible if the line of action of the resultant passes outside the body. But even so we can take some point lying on the line of action of the resultant, fix it (mentally) in the body and assume it to be the point of application of the resultant.

If a body acted upon by applied forces presses on a constraint, then the constraint in turn acts on the given body. *The force with which a constraint acts on a body preventing its displacement in any direction is termed the reaction force of the constraint.*

By the law of equality of action and reaction, the *reaction force of a constraint is equal to the magnitude of the force of pressure on the constraint and is opposite to that force in direction, that is, it is opposite to the direction in which the constraint prevents the displacement of the body.*

All forces acting on a body may be divided into active forces and the reaction forces of constraints. *Active forces include all forces that are not constraint reactions.*

Unlike active forces, the force of constraint reaction depends both on other forces acting on the given body and on the motion of the body and the nature of the imposed constraints. It exists only when the body, under the action of applied active forces, exerts a pressure on the given constraint. If there is no action on a constraint, there will be no constraint reaction (no force of reaction of a constraint).

The magnitude of the reaction force of a constraint is always unknown at first. The direction of that force is known when the given constraint can prevent the motion of a body only in one definite direction, otherwise the direction of the reaction force of a constraint is also unknown beforehand and it becomes apparent only when working the problem.

Problems involving the equilibrium of nonfree bodies are solved in statics on the basis of the following obvious circumstance: *every nonfree body may be regarded as a free body if one mentally removes the constraints and if their action on the body is replaced by the forces of reaction of the constraints (this is the principle of removability, or the axiom of constraints).*

Using this principle, it is possible to apply to a nonfree body the conditions of equilibrium established in statics for a free body. *All that needs to be done is to include in the forces acting on the body the reaction forces of the discarded constraints.*

Most engineering problems of statics consist precisely in determining the reaction forces of the constraints. If we know them, we will also know the forces of pressure on the constraints, that is, we will have sufficient data to calculate the strength of the structures under consideration.

We now consider how to determine the direction of reactions of some of the basic types of constraints.

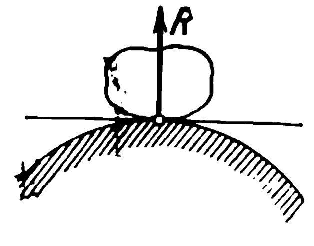


FIG. 7

1. *A smooth supporting surface*

A smooth surface is one whose friction generated by a body moving on it may be ignored.* Since a smooth supporting surface does not prevent a body from sliding along it, the *reaction* **R** *of the smooth surface* (Fig. 7) *is always directed along the common normal*** *to the surface of the body and the surface of the constraint at the point of contact.*

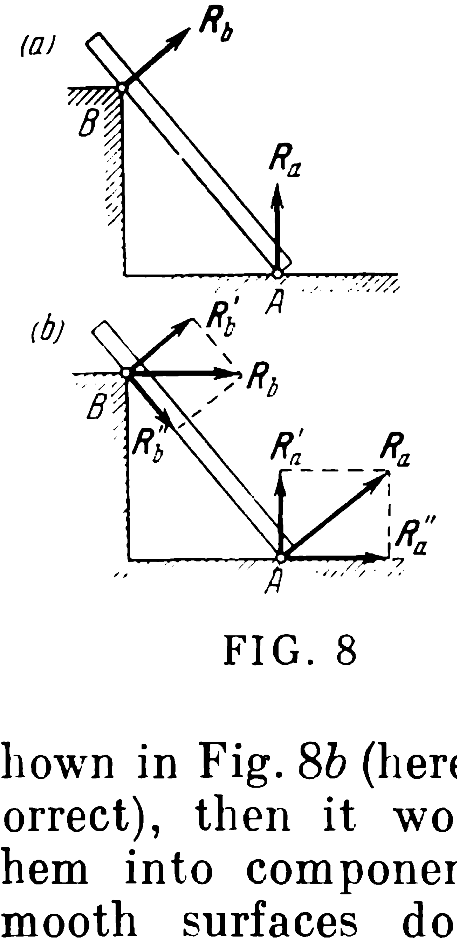


FIG. 8

If one of the contacting surfaces has a tip (Fig. 8), the reaction must be directed along the normal to the other surface. For example, to a smooth bar *AB* (Fig. 8*a*) resting at *A* on the floor and at *B* on a pole are applied reactions $\mathbf{R}_a$ of the floor and $\mathbf{R}_b$ of the pole (their directions are as indicated in the figure). If these reactions were not directed in that manner, say as shown in Fig. 8*b* (here the directions of the reactions are incorrect), then it would always be possible to decompose them into components as indicated in the figure. But since smooth surfaces do not prevent sliding along them, the components $\mathbf{R}_a''$ and $\mathbf{R}_b''$ cannot exist.

2. *A stationary cylindrical hinge or bearing*

A cylindrical hinge is a joining of two bodies by means of a pin (bolt) passing through an opening in the bodies.

---

* Constraints without friction belong in the class of *ideal* constraints, and for the time being we will confine ourselves to this type.

** The normal to a surface is the perpendicular (to the tangential plane) drawn through the point of tangency.

The diameter of the opening in the sleeve (Fig. 9) is slightly greater than the diameter of the pin. A body rigidly joined to a sleeve can only rotate about the axis of the hinge (the axis of the pin) that is perpendicular to the plane of the drawing.

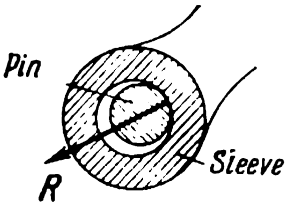


FIG. 9

One can often disregard the friction in the hinge (between the surfaces of the pin and the sleeve). In such a hinge (which is called ideal) there is nothing to prevent the sleeve from rotating about the axis of the pin or from moving along that axis. An ideal hinge merely prevents any motion of the sleeve in the direction of the normal to the surface of the sleeve and pin, and, hence, its reaction can only be directed along that normal (along the radius of the pin). But since the sleeve, depending on its position and the forces applied to it, can press on any point of the pin, it is impossible to state beforehand the direction of the reaction of the cylindrical hinge. The only assertion we can make (if friction in the hinge is disregarded) is that the *reaction of a stationary cylindrical hinge lies in the plane perpendicular to its axis and has a radial direction.*

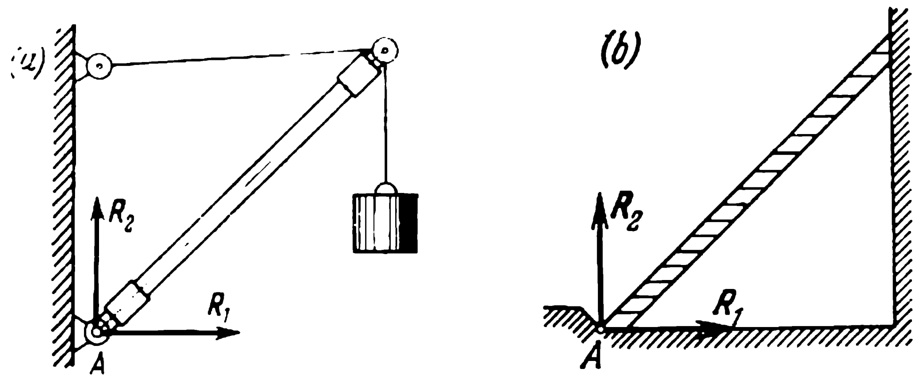


FIG. 10

To determine the constraint reaction in cases where its direction is indeterminate, it is very often useful to replace the desired reaction by several constituent reactions that are unknown in magnitude alone. For example, the reactions of the hinge A (Fig. 10*a*) or the support A (Fig. 10*b*) may conveniently be decomposed into a horizontal component $\mathbf{R}_1$ and a vertical component $\mathbf{R}_2$. In both cases we may assume that the constraint A (hinge or support) prevents the displacement of the body both in the horizontal and vertical direction.

After the magnitudes of each of the components of the reactions are found, we can find (if necessary) the total reaction of the constraint as the resultant of its components.

Note that by decomposing the reaction into components (in the given examples, into a horizontal component and a vertical component), one need not bother about the proper choice of directions along the chosen lines of their action. If actually one or the other component turns out to be in the opposite direction to that assumed, we will obtain (as we will see later on) a negative value for it.

A fixed-hinged support is frequently depicted schematically by two rods (Figs. 11*a* and 11*b*) joined at one end by a common hinge A, the axis of which will clearly be stationary. The reaction of such a support passes through the axis of the hinge but is not known either as to magnitude or direction and, hence, is described by two unknown elements.

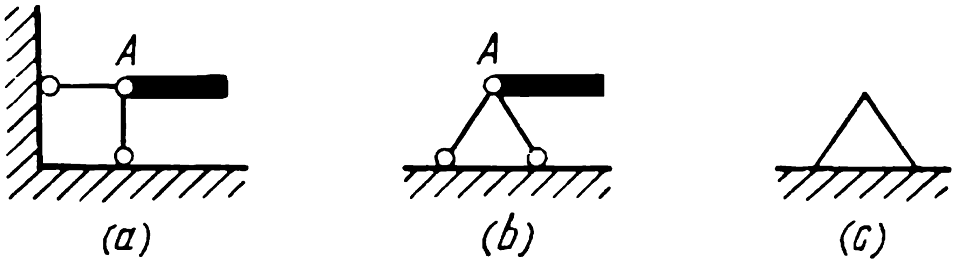


FIG. 11

Determining the magnitude and direction of the reaction of a fixed-hinged support may, as was indicated above, be replaced by determining the magnitudes of the two components of that reaction. The fact that the scheme of a fixed-hinged support has two rods indicates that two elements of the reaction of the given constraint are unknown.

According to the USSR State Standard, a fixed-hinged support for a rod is depicted as shown in Fig. 11*c*.

3. *Roller bearing*

Up to now we have been speaking of constraints accomplished by absolutely smooth surfaces. These constraints prevent the displacement of a body only in the direction normal to the surface and are characterized by a single normal reaction. A rough surface not only prevents displacement that upsets the constraint but also hampers displacement along the surface itself. Consequently, reaction of a rough surface has two components: one, the normal component to the surface, and the other one lying in the sliding plane (in the common tangential plane to the surface of the body and the supporting surface) and in the direction opposite that in which the body is being moved or any attempt is made to move it. The former component is termed the normal reaction, the latter goes by the name of forces of friction. Consequently, rough supporting surfaces differ in that the forces of friction must be taken into account.

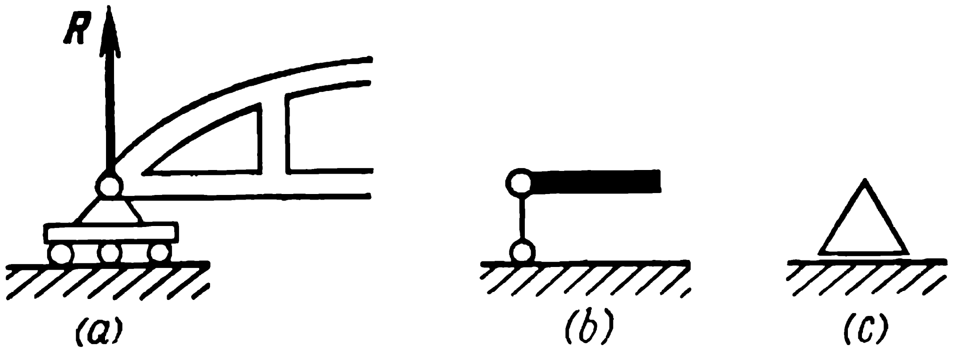


FIG. 12

How that is done will be discussed later on in the chapter on friction. For the present, let it be noted that although there is no such thing as an ideally smooth surface or, consequently, an ideal constraint in reality, in many cases friction may be disregarded and the constraints be regarded as ideal.

An instance is the frequently used *roller support* (Fig. 12*a*). The mobility of the roller is so great and, consequently, the friction is so small that it may be assumed the given constraint prevents only displacement perpendicular to the support plane; it is for this reason that it is *always characterized by a single normal reaction.*

A roller support does not prevent displacement of the hinge axis parallel to the support plane and constitutes a so-called movable-hinged support.

Such a support is frequently depicted schematically by a single rod with a hinge at one end (Fig. 12*b*). The reaction of such a support passes through the axis of the hinge and is directed along the normal to the supporting surface. The presence of a single rod in the support scheme indicates that only one element of reaction of the movable-hinged support is unknown, its magnitude.

According to the USSR State Standard, a movable-hinged support for a rod is depicted as shown in Fig. 12*c*.

4. *Thrust bearing*

A thrust bearing (Fig. 13) is a combination of a cylindrical hinge and a support plane. This constraint permits a shaft (cylinder) to rotate on its axis and to move along the axis, but only in one direction.

The reaction of a thrust bearing is made up of the reaction of a cylindrical bearing lying in the plane perpendicular to its axis (in the general case it may be decomposed into the components $\mathbf{R}_1$ and $\mathbf{R}_2$ shown in Fig. 13) and the normal reaction $\mathbf{R}_3$ of the plane of support.

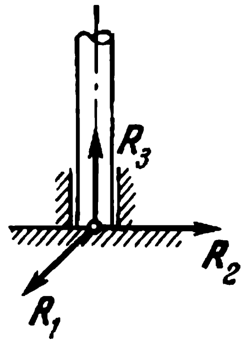


FIG. 13

5. *Flexible constraint*

Suppose a weight G is suspended at a point A on wires as shown in Fig. 14.

If the wires are inextensible, they prevent A from moving away from C in the direction CA and from B in the direction BA. Hence, the reactions* $\mathbf{T}_C$ and $\mathbf{T}_B$ of inextensible flexible wires are always in the direction along the wires towards their point of suspension.

Clearly, a flexible constraint can be in the form of a wire, thread, string, chain, cable, and so forth.

6. *Weightless rod*

The wires in Fig. 14 may be replaced, without changing the directions of the reactions, by rigid rods, if their weight is disregarded and if we assume the rods to be joined by ideal hinges. In the absence of friction, in ideal hinges the forces applied to the ends of the rods are always directed along the rods. These forces can only stretch or compress the rods. Indeed, as we know, an ideal hinge is characterized by only one reaction normal to the axis of the hinge. But two forces applied to the ends of a rod can balance only when they are equal in magnitude and opposite in direction along a straight line. Hence, the *reaction of a weightless and hinge-connected rod is directed along the line joining the centres of the hinges.*

---

* It is customary to denote the reactions of flexible constraints by the letter **T**.

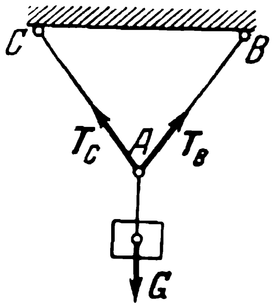


FIG. 14

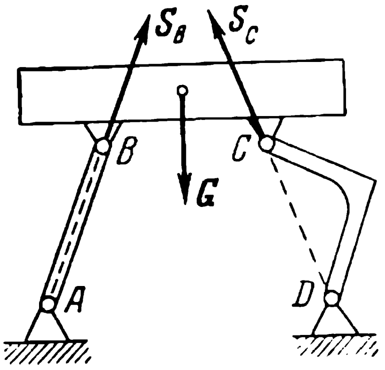


FIG. 15

Clearly, the directions are the same for the reactions of hinged rods supporting a body (Fig. 15). *The algebraic value of the force acting along the rod and extending or compressing it is termed the force inside the rod.* Ordinarily, the force is positive in the case of tension and negative under compression.*

In the sequel we will consider first the composition, decomposition and equilibrium of forces in those cases where the lines of action of the forces lie in one plane, and then in cases where they do not lie in the same plane. We start the study of a plane system with the so-called system of converging forces.

---

* The reactions of rods are often denoted by the letter S.

CHAPTER II

PLANE SYSTEM OF CONVERGING FORCES

Sec. 8. Converging forces. The composition of two forces applied at a single point

A system of converging forces is a system of forces whose lines of action intersect in one point. According to the first corollary to the axioms of statics, the action of the system on an absolutely rigid body does not change if we translate all forces of the given system along their lines of action to the common point of intersection of the lines. It is thus possible to replace any system of converging forces by an equivalent system of forces applied at a single point.

In this chapter we will discuss only such a system of converging forces whose lines of action lie in a single plane, that is, the so-called *plane system of converging forces.*

The problem of the composition of two forces applied to one point is solved graphically in an extremely simple fashion. Referring to Fig. 16, suppose two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are applied at point A of a rigid body. On the basis of the third axiom of statics (the rule of the parallelogram of forces), the resultant $\mathbf{R}_P$ of the given forces is applied to the same point A and is depicted in magnitude and direction by the diagonal of a parallelogram constructed on those forces as sides.

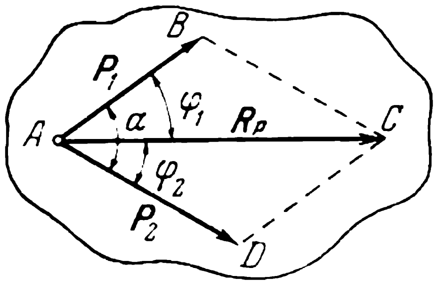


FIG. 16

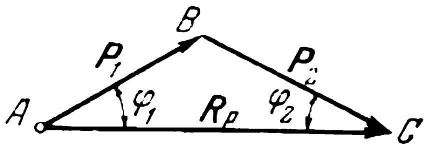


FIG. 17

To find the resultant there is no need to construct the entire parallelogram $ABCD$, it suffices to construct only one of the triangles ABC or ADC. To construct one of them (say ABC, Fig. 17), from the terminus of the vector of one force $\mathbf{P}_1$ draw the vector $\overrightarrow{BC}$ equal to the vector of the second force $\mathbf{P}_2$. The closing side AC of the triangle ABC depicts in magnitude and direction the resultant of two given converging forces. It only remains to measure its length in the given scale and to determine the angles φ_1 and φ_2. The triangle ABC (or ADC) is called the *force triangle* (or *triangle of forces*), and the method of composition of two forces is termed the *triangle rule*.

The resultant of two forces applied to a single point is readily found by calculation. To do this, construct the same force triangle ABC (or ADC) but do not make any effort at exactness because it will merely serve as an illustration to the solution. Denote the angle between the given forces $\mathbf{P}_1$ and $\mathbf{P}_2$ by α, and the angles that the resultant forms with these forces by φ_1 and φ_2 respectively (Fig. 16). The sides of the triangle ABC are, to some scale, the numerical values (magnitudes) of the forces and therefore, by the familiar trigonometric cosine law, we have

$$R_P^2 = P_1^2 + P_2^2 - 2P_1P_2 \cos \angle ABC.$$

Besides, $\angle ABC = 180° - \alpha$ and $\cos \angle ABC = \cos (180° - \alpha) = -\cos \alpha$. Substituting this value into the equation above, we have

$$R_P^2 = P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha$$

whence the magnitude of the resultant of two converging forces is

$$R_P = \sqrt{P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha}. \qquad (1)$$

We take the plus sign in front of the root because the magnitude of a vector is always positive.

Now let us determine the direction of the resultant.

By the law of sines, which states that in any triangle the sides are proportional to the sines of the opposite angles, we obtain, from the same triangle, ABC,

$$\frac{AB}{\sin \angle BCA} = \frac{BC}{\sin \angle BAC} = \frac{AC}{\sin \angle ABC}.$$

But $\angle BCA = \angle CAD = \varphi_2$, $\angle BAC = \varphi_1$, $\angle ABC = 180° - \alpha$ and $\sin(180° - \alpha) = \sin\alpha$.

Now, the sides of the triangle are proportional to the magnitudes of the forces. Consequently, the law of sines for the given force triangle yields the following relationship:

$$\frac{P_1}{\sin\varphi_2} = \frac{P_2}{\sin\varphi_1} = \frac{R_P}{\sin\alpha}. \tag{2}$$

Formula (2) permits finding the sines of the angles between the resultant and the component forces, and hence, the angles as well.

Let us now consider some special cases that are frequently encountered in practice.

(1) If the angle between the given forces $\alpha = 90°$, then

$$\cos\alpha = \cos 90° = 0$$

and the magnitude of the resultant is

$$R_P = \sqrt{P_1^2 + P_2^2}. \tag{3}$$

(2) If the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are along a single straight line and in the same direction, then the angle between them is $\alpha = 0$ and $\cos\alpha = 1$. The magnitude of the resultant is

$$R_P = \sqrt{P_1^2 + P_2^2 + 2P_1P_2} = P_1 + P_2. \tag{4}$$

If the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are along a single straight line in opposite directions, the angle between them is $\alpha = 180°$, $\cos\alpha = -1$ and $R_P = \sqrt{P_1^2 + P_2^2 - 2P_1P_2} = P_1 - P_2$ if $P_1 > P_2$, and $R_P = P_2 - P_1$ if $P_1 < P_2$.

Consequently, *the resultant of two forces acting on a body along a single straight line is directed along that line and its magnitude is equal to the sum of the magnitudes of the component forces if the forces are in the same direction, and is equal to the absolute value of the difference if the forces are in opposite directions.*

Problem 1. Two persons are pulling ropes attached to a ring at point A (Fig. 18) at right angles, one with a force $P_1 = 120$ N and the other with a force $P_2 = 90$ N. What force must be applied by a third person for the ring to remain fixed?

Solution. For the ring to remain in position, the third person must act on it with a force that balances the action of the other two persons. Since the balancing force and the resultant force are equal in magnitude and opposite in direction (the second corollary to the axioms of statics), the problem reduces to determining the magnitude and direction of the resultant of two given converging forces.

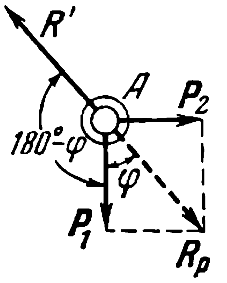


FIG. 18

According to formula (3) the magnitude of the resultant is

$$R_P = \sqrt{90^2+120^2} = 150\text{ N}.$$

The angle formed by the resultant and the direction of the force of 120 N is determined from the right-angle force triangle shown in Fig. 18:

$$\tan\varphi = \frac{90}{120} = 0.75, \quad \varphi = 36^\circ 50'.$$

Hence the third person must pull the ring with a force R' of 150 N in the direction that forms with the given force of 120 N an angle of $180^\circ - \varphi$, that is, an angle of 143°10′.

Sec. 9. Decomposing a force into two converging components

To decompose (resolve) a force into two or more component forces means to find a system of two or several forces that will produce on a body the same effect as the single given force. In other words, to decompose a force into, say, two components means to find two forces whose resultant is equal to the given force.

Clearly, if it is possible to replace two converging forces $\mathbf{P}_1$ and $\mathbf{P}_2$ by a single force (the resultant) $\mathbf{P}$, then, conversely, it is possible to replace the action on a body of a single given force $\mathbf{P}$ by the actions of two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ converging on the line of action of the given force. The only requirement here is that the given force $\mathbf{P}$ be depicted in magnitude and direction by the diagonal of a parallelogram whose sides are the segments depicting the forces $\mathbf{P}_1$ and $\mathbf{P}_2$.

Since a given diagonal yields an infinitude of parallelograms, it is necessary to specify supplementary conditions in order to make the solution determinate.

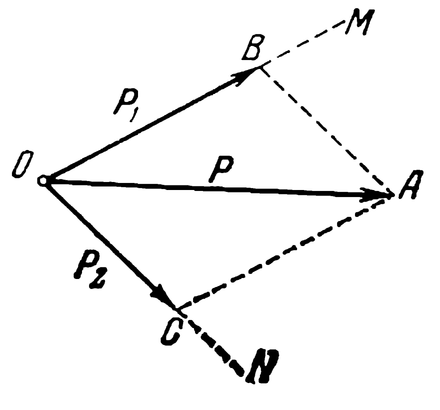


FIG. 19

Such supplementary conditions may be: (1) specifying two directions in which the components must act; (2) specifying the magnitude and direction of one of the component forces; (3) specifying the magnitudes of both component forces; (4) specifying the magnitude of one of the components and the direction of the other.

We now consider the first case, which is the one most frequently encountered. It is required to resolve (decompose) the force **P** into two converging forces whose directions *OM* and *ON* are given (Fig. 19).*

To solve the problem, draw lines *AB* and *AC* from the terminus *A* of the vector of the force **P**; these lines are to be parallel to the straight lines *ON* and *OM*. The result is a parallelogram *OBAC* for which the force **P** is a diagonal. The vectors $\overrightarrow{OB}$ and $\overrightarrow{OC}$ yield, in the same scale as the given force **P**, the desired components $\mathbf{P}_1$ and $\mathbf{P}_2$.

Since the sum of the nonparallel sides of a parallelogram is always greater than each of the diagonals, the sum of the magnitudes of the two converging components into which the given force was resolved is always greater than the magnitude of that force.

The greater the angle between the directions of the component forces, the greater their numerical value. For a sufficiently large angle between the components, the magnitude of each may be even greater than the magnitude of the force being resolved.

Referring to Fig. 20, let a load weighing *G* be suspended at the midpoint *C* of a rope, the ends of which *A* and *B* are fixed at a given height. In order to determine the tension of each branch of the rope, we resolve the force G, via the parallelogram law, into components $\mathbf{T}_1$ and $\mathbf{T}_2$ that are directed along the lines *BC* and *AC*. From the figure it is evident that the magnitudes T_1 and T_2 of the forces that

---

* Here and henceforth we do not give the outline of the body to which a force **P** is applied at point *O*, for this plays no part in solving the problem.

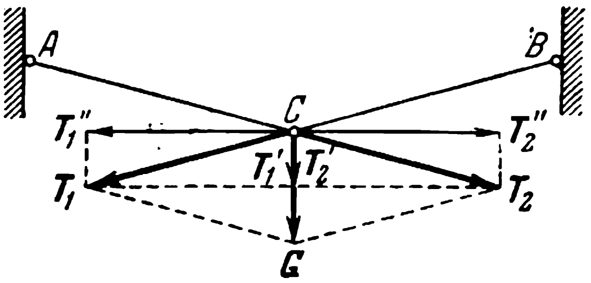


FIG. 20

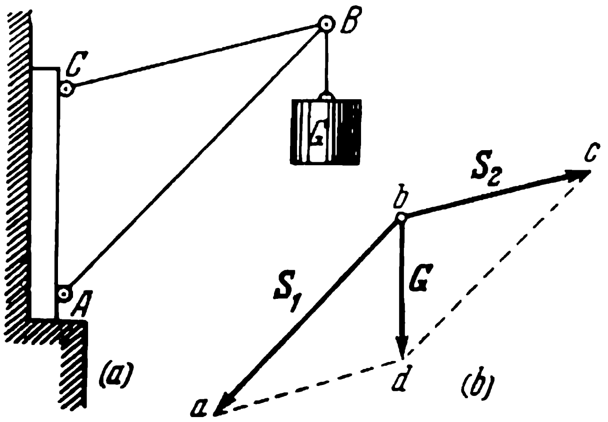


FIG. 21

stretch the branches of the rope and are the components of the force G of gravity of the load exceed the magnitude of that force. This is explained by the fact that the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ partially cancel each other. Indeed, each of these forces may, in turn, be resolved into a horizontal and vertical component. Clearly, the components $\mathbf{T}_1''$ and $\mathbf{T}_2''$ that are equal in magnitude and opposite in direction will cancel out. That leaves the magnitudes of the vertical components $\mathbf{T}_1'$ and $\mathbf{T}_2'$ whose sum is exactly equal to the magnitude of the vertical force G.

Problem 2. Replace a force **P** equal in magnitude to 96 N by two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ that form angles of 30° and 45° with the force **P**.

Solution. Using formula (2) we get

$$\frac{P_1}{\sin\varphi_2}=\frac{P_2}{\sin\varphi_1}=\frac{R_P}{\sin\alpha}.$$

We have $\varphi_1=30°$, $\varphi_2=45°$, $\alpha=\varphi_1+\varphi_2=75°$ and $R_P=P=$ $=96$ N, whence

$$P_1=\frac{R\sin\varphi_2}{\sin\alpha}=\frac{96\times0.707}{0.966}\approx70.3\text{ N},$$

$$P_2=\frac{R\sin\varphi_1}{\sin\alpha}=\frac{96\times0.5}{0.966}\approx49.7\text{ N},$$

Problem 3. A crane* *ABC* (Fig. 21*a*) holds suspended a load *G* of 10 kN. Find the forces S_1 and S_2 in the rods *AB* and *BC*. Dimensions of the rods: $AB = 3.8$ m, $BC = 2.6$ m, and the distance $AC = 2$ m.

Solution. Resolve the force G into two components along the rods *BC* and *AB*. To do this, construct a parallelogram *abcd* (Fig. 21*b*) with sides parallel to the rods and with a diagonal *bd* directed vertically downwards (in accordance with the direction of the force G of gravity of the load) and numerically equal (in some scale) to the magnitude of the force G.

The lengths of the sides *ba* and *bc* of the parallelogram yield (in the same scale that the magnitude of G was laid off) the absolute values of the desired forces S_1 and S_2.

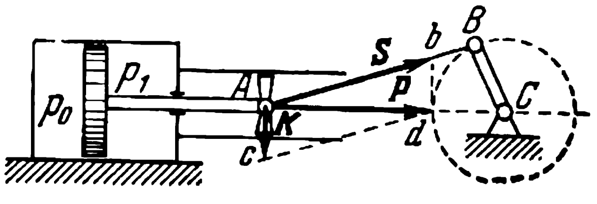


FIG. 22

To calculate these forces it is best here to make use of geometric reasoning. The triangles *CBA* and *bcd* (Figs. 21*a* and 21*b*) are similar since the sides of these triangles are parallel. From the similarity of the triangles we find

$$\frac{bc}{BC} = \frac{cd}{AB} = \frac{bd}{AC}.$$

Or, taking into account that the sides of triangle *bcd* are proportional to the magnitudes of the forces they depict, we obtain

$$\frac{|S_1|}{BC} = \frac{|S_2|}{AB} = \frac{G}{AC},$$

whence

$$|S_2| = \frac{BC}{AC}G = \frac{2.6}{2} \times 10 = 13\,\text{kN}$$

and

$$|S_1| = \frac{AB}{AC}G = \frac{3.8}{2} \times 10 = 19\,\text{kN}.$$

If we take into account the directions of the forces S_1 and S_2, it will be clear that the rod *AB* experiences compression and so $S_1 = -19$ kN, and that the rod *BC* experiences tension and so $S_2 = +13$ kN.

Problem 4. The diameter of a piston in a steam engine (Fig. 22) is $d = 40$ cm, the length of the connecting rod $AB = 1.8$ m, the length of the crank $BC = 0.36$ m, the steam pressure in the cylinder behind the piston is $p_0 = 700$ kPa*, in front of the piston $p_1 = 100$ kPa.

---

* A hoisting machine for moving loads vertically and horizontally.

Determine the force **S** imparted to the connecting rod and the pressure **K** of the slide bar *A* on the guide bars for that position of the piston when

$$\angle ABC = 90^\circ.$$

Solution. The force of pressure of the steam on the piston is

$$P=(p_0-p_1)\,\frac{\pi d^2}{4}=(700-100)\,\frac{3.14\times 0.4^2}{4}=75.4\,\text{kN}$$

This pressure on the piston is transmitted via a rod to the point *A* of the slide bar. Let us translate the force **P** along the line of its action to the point *A* by laying off $\overrightarrow{Ad}$ = **P** (to some scale), and resolve it into two components: the force **S** = $\overrightarrow{Ab}$ transmitted to the connecting rod and directed along its axis, and a vertical force **K** = $\overrightarrow{Ac}$ with which the slide bar exerts pressure on the lower guide bar. The right triangles *ABC* and *Adc* are similar because they have equal acute angles *BAC* and *Adc*.

Setting up the ratios of similar sides, we obtain

$$\frac{Ac}{BC}=\frac{cd}{AC}=\frac{Ad}{AB}$$

or, taking into account that $Ab = cd$ and that the sides of the force triangle are proportional to the magnitudes of the corresponding forces, we obtain

$$\frac{K}{BC}=\frac{S}{AC}=\frac{P}{AB}.$$

From this the force of pressure of the slide bar *A* on the guide bar is

$$K=P\,\frac{BC}{AB}=75.4\times\frac{0.36}{1.8}=15.1\,\text{kN}.$$

The force transmitted by the piston to the connecting rod,

$$S=P\,\frac{AC}{AB}=P\,\frac{\sqrt{AB^2+BC^2}}{AB}=75.4\times\frac{\sqrt{3.37}}{1.8}\approx 77\,\text{kN}.$$

---

* In the SI system of units, the unit of pressure is the pascal (abbreviated Pa), which is equal to the pressure resulting from a force of one newton acting uniformly over an area of one square meter. A unit of pressure still in use (in the engineering system of units) is the unit of pressure equal to 1 kgf/cm^2 ≈ 98.1 kPa, where kPa is the kilopascal. The SI unit of pressure, pascal, was named in honour of the prominent French mathematician and physicist Pascal (1623-1662).

Sec. 10. Polygon of forces

Suppose it is required to add four converging forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$.

Referring to Fig. 23*a*, let us depict the given forces to some arbitrary scale by vectors applied to the point A of intersection of their lines of action.

We will add the forces in succession using the familiar force-triangle rule for the composition of two converging forces.

We first use this rule to add two of the given forces, say the forces $\mathbf{P}_1$ and $\mathbf{P}_2$; to do this, draw a vector $\overrightarrow{BC} = \mathbf{P}_2$ from the terminus of force $\mathbf{P}_1$. The resultant $\mathbf{R}_{12}$ of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is depicted to the chosen scale by the closing side of the triangle, that is, by the vector $\overrightarrow{AC}$. Now use the same rule again to combine the forces $\mathbf{R}_{12}$ and $\mathbf{P}_3$; to do this, draw a vector $\overrightarrow{CD} = \mathbf{P}_3$ from point C and join A and D. The vector $\overrightarrow{AD} = \mathbf{R}_{123}$ constitutes, in the chosen scale, the resultant of the forces $\mathbf{R}_{12}$ (which in turn is the resultant of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$) and $\mathbf{P}_3$, which is to say it replaces the action of the forces $\mathbf{P}_1$, $\mathbf{P}_2$ $\mathbf{P}_3$.

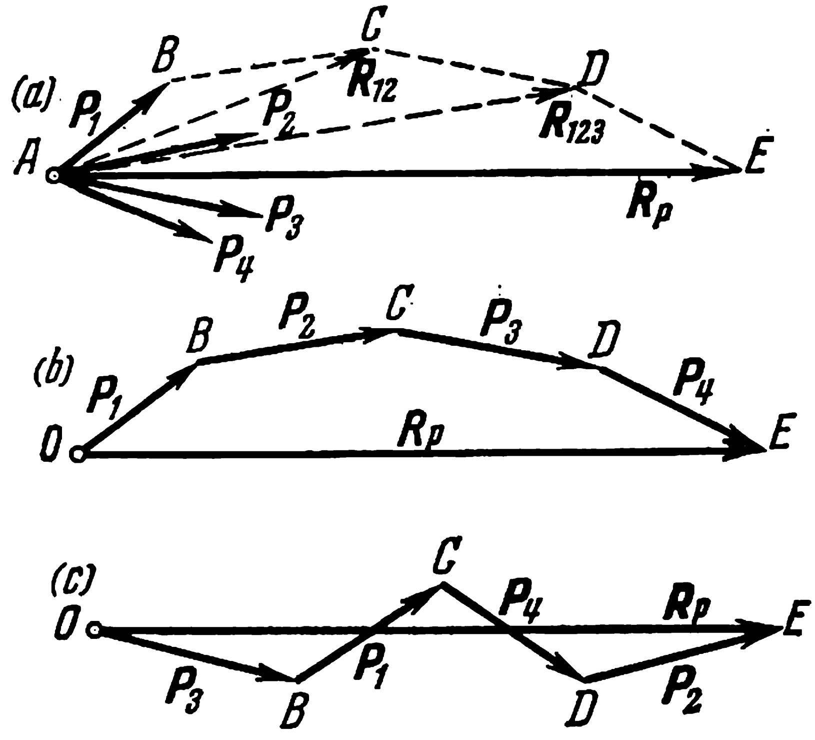


FIG. 23

Continuing the adding, we combine the forces $\mathbf{R}_{123}$ and $\mathbf{P}_4$. From the point D draw the vector $\overrightarrow{DE} = \mathbf{P}_4$ and join points A and E with a straight line. The vector $\overrightarrow{AE} = \mathbf{R}_P$, which in the chosen scale is the resultant of the forces $\mathbf{R}_{123}$ and $\mathbf{P}_4$, clearly is the resultant of the entire given system of converging forces.

It is ordinarily more convenient to carry out the process of constructing the resultant of a system of converging forces in a somewhat different manner (Fig. 23*b*). In the plane of action of the forces, choose an arbitrary point O and from it lay off a vector $\overrightarrow{OB}$ equal, in the chosen scale, to the force $\mathbf{P}_1$; from its terminus (point B) draw a vector $\overrightarrow{BC}$ equal to the force $\mathbf{P}_2$; from the terminus of that vector (point C) draw a vector $\overrightarrow{CD}$ equal to the force $\mathbf{P}_3$, and so forth, each time placing the origin of the next vector at the terminus of the preceding one until all forces are exhausted.

The resulting *polygon OBCDE*, *whose sides in the chosen scale are equal to the given forces and are in the same direction, is termed a polygon of forces* (*force polygon*). In a force polygon, the arrows follow in succession one after the other, and not in opposing directions.

The *closing side* $\overrightarrow{OE}$ *of the force polygon is directed from the origin of the first force to the terminus of the last force and in the chosen scale depicts the resultant of the given system of converging forces* both in magnitude and direction. This is quite evident both from the construction and from a comparison of Figs. 23*a* and 23*b*.

In order to determine the line of action of the resultant, draw through the common point of the lines of action of the converging forces a straight line parallel to the closing side of the force polygon.

This rule for the composition of converging forces via a polygon is the general rule for adding any kinds of vectors and is called the rule of geometric addition. It holds true for any number of vectors being added (in the given case, for any number of converging forces).

As a formula, the geometric addition of forces is written thus:

$$\mathbf{R}_{\mathrm{pr}}=\mathbf{P}_1+\mathbf{P}_2+\ \dots\ +\mathbf{P}_n.$$

The geometric sum of all forces of a given system is termed the principal vector of the system.

All the summands of the given sum are of one and the same form and differ from one another solely in the values of the index. This sum, like any other similar sum, may be compactly written by means of the sigma notation (the capital Greek letter sigma):

$$\mathbf{R}_{\mathrm{pr}}=\sum_{k=1}^{k=n}\mathbf{P}_k.$$

The symbol $\sum\limits_{k=1}^{k=n}$ signifies that the indicated expressions to the right are to be added, the index k running through all integral values from $k=1$ under the sign to $k=n$ above the sign. To further simplify notation, the range of variation of the index is sometimes dropped, leaving the symbol Σ and assuming the sum of n terms to be added, $k=1, 2, 3, \dots, n$. We then have*

$$\mathbf{R}_{\mathrm{pr}}=\sum\mathbf{P}_k. \tag{5}$$

From the above described rule for finding the resultant via a force polygon it follows that the *resultant* $\mathbf{R}_p$ *of a system of converging forces is equal to their principal vector* $\mathbf{R}_{\mathrm{pr}}$. The line of action of the resultant passes through the common point of intersection of the lines of action of the component forces.

Generally, the concept of the principal vector of a given system of forces (that is, of a geometric sum of forces) is not equivalent to the concept of their resultant. When the lines of action of the component forces do not intersect in a single point, the geometric sum of the forces, as we shall see later on, does not fully determine their resultant. Even more, in certain cases the system may not even have a resultant altogether, whereas the principal vector may be found for any system of forces.

---

* This form of notation will be used throughout the book in similar cases.

Note that the order in which a vectorial polygon is constructed may be changed and the closing side of the polygon will not be altered in the process (Fig. 23*c*) either in magnitude or in direction.

A geometric sum is not altered due to a change in the order of the summands.

Problem 5. Four forces lying in a single vertical plane and forming between them the angles 75°, 45° and 120° are applied to a point of a body. The first force is directed upwards and to the right at an angle of 60° to the vertical. The magnitudes of the forces are: $P_1 = 150$ N, $P_2 = 200$ N, $P_3 = 250$ N, $P_4 = 120$ N.

Determine (graphically) the magnitude and direction of the resultant.

Solution. Chosing some scale of the forces, for example $\mu_P = 10$ N/mm, lay off on the specified directions line segments depicting

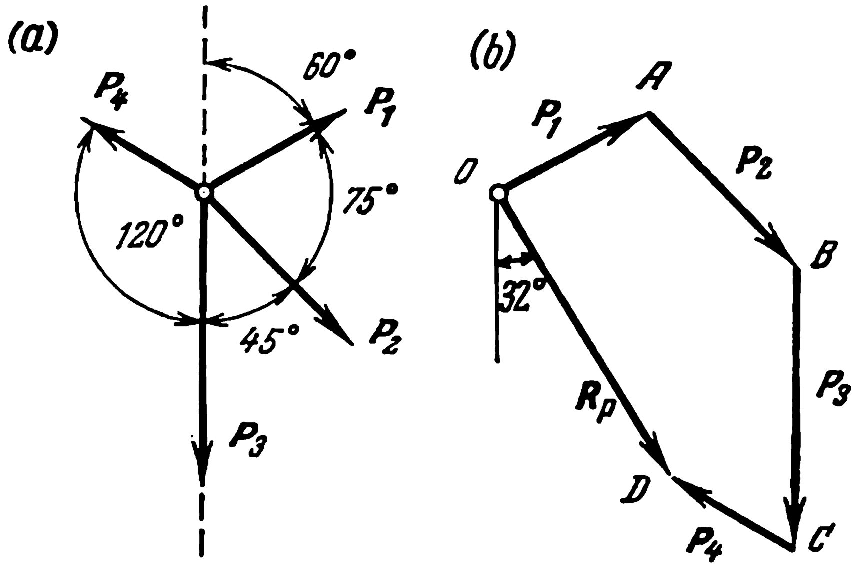


FIG. 24

the appropriate vectors of the forces (Fig. 24*a*). From an arbitrary point O draw a line segment OA equal to the vector of the given force $\mathbf{P}_1$ (Fig. 24*b*). From the end point A of that segment draw AB equal to the vector of another given force $\mathbf{P}_2$, and so on. The closing side OD of the resulting force polygon depicts, to the given scale, the principal vector of the given system of converging forces equal to their resultant ($\mathbf{R}_{\text{pr}} = \mathbf{R}_P$).

The closing side $OD = 30$ mm and so the magnitude of the resultant $R_P = \mu_P \times OD = 10 \times 30 = 300$ N. Measuring the angle between the vertical and the direction of the closing side, we find that the resultant $\mathbf{R}_P$ is directed to the right and downwards at an angle of 32° to the vertical.

Problem 6. The tractive force of a steam tugboat is $P = 32$ kN. How many barges can the boat tug at a velocity constant in magnitude and direction if the water resistance to the motion of the tugboat is $P_1 = 8$ kN and the resistance to the motion of each barge is $P_2 = 6$ kN. Also determine the tension of the cable between the tugboat and the first barge, between the first barge and the second barge, and so on.

Solution. Since the tugboat is hauling a caravan of barges at a constant speed, its tractive force is a balancing force for the forces of resistance of the water to the moving tugboat and barges, and, hence, is equal in magnitude to the resultant of those forces and is in the same direction along a straight line, $P = P_1 + P_2 n$, and so the number of barges

$$n = \frac{P - P_1}{P_2} = \frac{32 - 8}{6} = 4.$$

The tension of the rope between the tugboat and the first barge is equal to the water resistance to the motion of the four barges: $T_1 = P_2 \times 4 = 6 \times 4 = 24$ kN. The tension of the rope between the first barge and the second barge is equal to the water resistance to the motion of three barges: $T_2 = P_2 \times 3 = 18$ kN. The tension of the rope between the third and fourth (last) barge is $T_4 = P_2 = 6$ kN.

Sec. 11. The projection of a vector on an axis. Determining a vector from its projection

An analytic determination of the resultant of a system of converging forces (that is, determining the magnitude and direction of the desired vector by calculation) is based on the use of the method of projections, to which we now proceed.

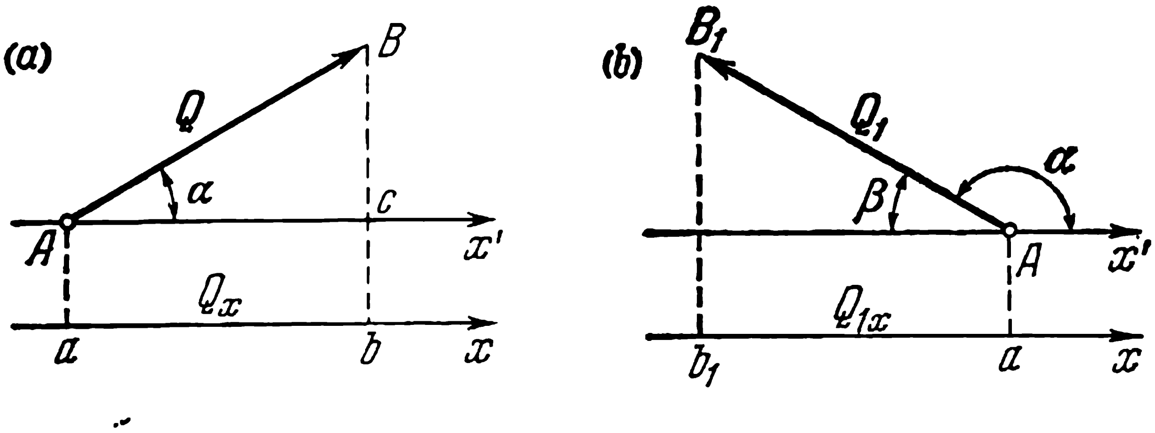


FIG. 25

A straight line of unbounded length given a specified direction is termed an axis. Take (see Fig. 25*a*) the vector $\overrightarrow{AB} = \mathbf{Q}$

and the x-axis (both lie in the plane of the drawing). For the axis we give the direction indicated by the arrow. Now drop perpendiculars from the origin A and the terminus B of the vector onto the axis.

The feet of the perpendiculars dropped from the given points onto the axis are called the projections of those points on the given axis.

The length of the segment of the axis lying between the projections on the axis of the origin and terminus of the given vector is called the projection of the vector on the given axis.

The projection of a vector on an axis is denoted by the same letter (lightface) as the vector itself, with the axis of projection indicated as a lower-case subscript. Thus, to illustrate, the projection of the vector **Q** on the x-axis is denoted as Q_x.

The projection of a vector on an axis is regarded as positive if its direction coincides with the accepted (positive) direction of the axis, and is negative otherwise. From Figs. 25*a* and 25*b* it will readily be seen that the projection of the vector on the axis is positive when the vector forms an acute angle with the direction of the axis of projections (Fig. 25*a*), and is negative when that angle is obtuse (Fig. 25*b*). Consequently, $Q_x = \pm ab$, the sign being taken according to the above rule.

Note that the *projection of a vector on an axis is* not a vector quantity but a *scalar algebraic quantity* since it is fully defined by its sign and the numerical value of the appropriate line segment on the projection axis.

Instead of projecting a vector on a given axis, it appears to be more convenient to project it on an axis parallel to the given one and in the same direction but passing through the origin of the vector. It is clear that the *projections of a vector on two parallel and similarly directed axes are equal*, as segments of parallel lines between parallels.

Draw through the origin A of the vector **Q** an axis x' (Fig. 25*a*) parallel to the given axis x and in the same direction. Denote by α the angle formed by the vector with the positive direction of the projection axis. Then, from the right triangle ABc we obtain

$$ab = Ac = AB\cos\alpha \quad \text{or} \quad Q_x = Q\cos\alpha. \qquad (6)$$

The projection of the vector on the axis is equal to the magnitude of the vector multiplied by the cosine of the angle between the vector and the positive direction of the projection axis.

This equality in all cases determines not only the numerical value of the projection but also its sign. In the case at hand (Fig. 25*a*), the angle α is an acute angle and so its cosine is positive and the projection is positive. In the case of a negative projection, the angle α between the vector and the positive direction of the projection axis will always be obtuse. But for an obtuse angle (Fig. 25*b*), $\cos \alpha = \cos (180^\circ - \beta) = -\cos \beta$ and in that case also $Q_{1x} = -ab_1 = -AB_1 \cos \beta = AB_1 \cos \alpha$.

In order to avoid trigonometric functions of obtuse angles when calculating projections, it is simpler to multiply straightway the magnitude of the vector being projected by the cosine of the acute angle between the vector and the projection axis, and only then adjoin to the projection the plus sign if the angle between the vector and the positive direction of the projection axis is acute, and the minus sign if that angle is obtuse.

We now consider two special cases.

(1) The vector is parallel to the projection axis, that is, $\alpha = 0$ or $\alpha = 180^\circ$ depending on which direction (positive or negative) of the projection axis coincides with the direction of the vector. In that case, $\cos \alpha = \pm 1$ and $Q_x = \pm Q$.

Consequently, *the projection of a vector on an axis parallel to it is equal to the magnitude of the vector taken with the plus or minus sign depending on the direction of the vector.*

(2) The vector is perpendicular to the projection axis, that is, $\alpha = 90^\circ$. Here, $\cos \alpha = 0$ and $Q_x = 0$.

Hence, *the projection of a vector on an axis perpendicular to it is equal to zero.*

If a vector and an axis are given, the projection of the vector is determined uniquely. However, specifying only the projection of a vector does not yet determine the vector itself, since different vectors can have the same projections on the same axis (Fig. 26). To determine a vector requires knowing at least its projections on two nonparallel axes in the plane of which the given vector lies. It is more convenient to have mutually perpendicular axes (the axes of a rectangular Cartesian system of coordinates in the plane).

In that case, as illustrated in Fig. 27, the vector **Q** is the diagonal of a rectangle whose sides are numerically equal to the projections of the vector on the coordinate axes, whence it follows that the magnitude of the vector is

$$Q=\sqrt{Q_x^2+Q_y^2}. \qquad (7)$$

The magnitude of a vector is equal to the square root of the sum of the squares of its projections on two mutually perpendicular axes in the plane of which the given vector lies.

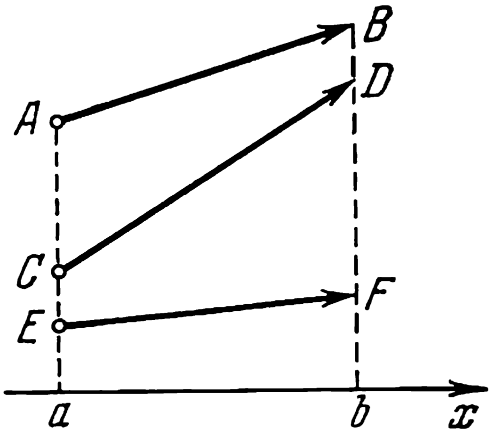


FIG. 26

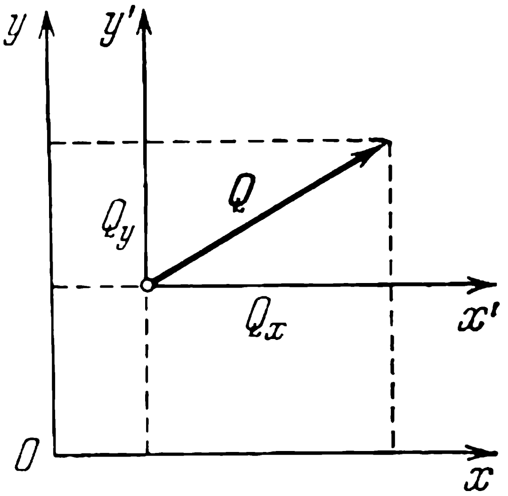


FIG. 27

In formula (7), one has to take the principal root because the magnitude of a vector is a positive number.

The direction of the vector is determined from the equality $Q_x = Q\cos\widehat{(\mathbf{Q}, x)}$ and $Q_y = Q\cos\widehat{(\mathbf{Q}, y)}$, where $\widehat{(\mathbf{Q}, x)}$ and $\widehat{(\mathbf{Q}, y)}$ are angles formed by the vector with the positive directions of the appropriate axes. From this, we have

$$\cos\widehat{(\mathbf{Q}, x)}=\frac{Q_x}{Q}, \quad \cos\widehat{(\mathbf{Q}, y)}=\frac{Q_y}{Q}. \qquad (8)$$

The cosine of the angle between the vector and the positive direction of the projection axis is called the direction cosine. It is equal to the ratio of the appropriate projection of the vector to the magnitude of the vector.

Formula (7) defines the magnitude of a vector, whereas formulas (8) make it possible to compute the cosines of the angles between the vector and the coordinate axes and, hence, to find the angles themselves that determine the direction of the vector.

Thus any vector is fully determined by its projections on the coordinate axes (by two projections when the vector and the axes lie in the same plane).

Sec. 12. The projection of a geometric sum of vectors on an axis

Theorem. *The projection of a geometric sum of vectors on some axis is equal to the algebraic sum of the projections of the component vectors on the same axis.*

Suppose we have several vectors, for instance, four vectors Q_1, Q_2, Q_3, Q_4 (to simplify the drawing we take the vectors in the same plane, but the theorem remains valid in the general case as well).

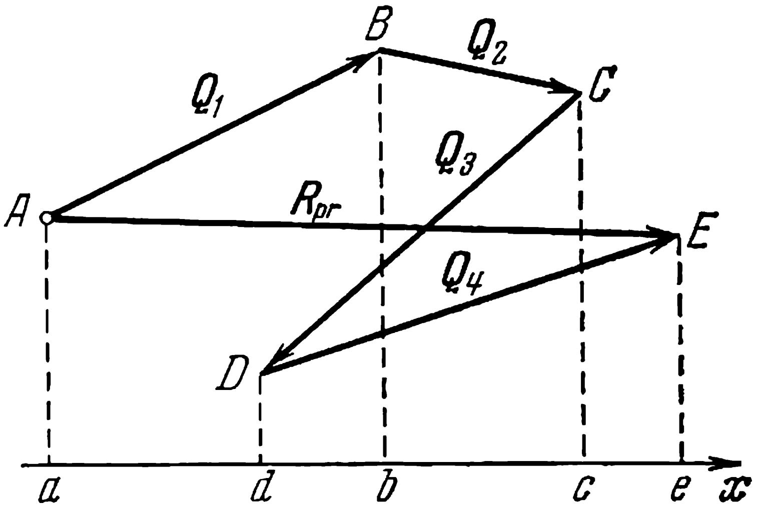


FIG. 28

By the rule of composition of vectors, the geometric sum of the given vectors is $\overrightarrow{AE} = \mathbf{R}_{\text{pr}}$, which is the closing side of the vector polygon $ABCDE$ whose sides are the component vectors (Fig. 28).

Projecting the vectors on the x-axis (Fig. 28), we obtain

$$Q_{1x} = ab, \quad Q_{2x} = bc, \quad Q_{3x} = -cd, \quad Q_{4x} = de, \quad R_{\text{pr}\,x} = ae.$$

From the drawing it is evident that $ae = ab + bc - cd + de$, that is,

$$R_{\mathrm{pr}\,x} = Q_{1x} + Q_{2x} + Q_{3x} + Q_{4x} = \sum Q_{kx}, \tag{9}$$

which is what we set out to prove.

The given theorem holds true for any vectors and any number of vectors.

Sec. 13. An analytic determination of the resultant of a plane system of converging forces

We already know that the resultant of a system of converging forces is equal to their geometric sum. At the same time, from Sec. 12 we know that the projection of a geometric sum of vectors on any axis is equal to the algebraic sum of the projections of the component vectors on the same axis.

Since this proposition holds true for all vectors, it follows that the *projection of the resultant of a system of converging forces on some axis is equal to the algebraic sum of the projections of the component forces on that axis.*

Denoting the projections of the resultant $\mathbf{R}_P$ on the coordinate axes x and y by R_{Px} and R_{Py}, and the projections of the component forces on the same axes by the capital letters X and Y with appropriate subscripts, as is ordinarily done, we have

$$\left.\begin{aligned} R_{Px} &= X_1 + X_2 + X_3 + \ldots = \sum X_k, \\ R_{Py} &= Y_1 + Y_2 + Y_3 + \ldots = \sum Y_k. \end{aligned}\right\} \tag{10}$$

If the projections of some force on two mutually perpendicular axes are known, in the plane of which the vector of the given force lies,* then the magnitude and direction can be found from (7) and (8). The magnitude of the resultant of a plane system of converging forces is given by the formula

$$R_P = \sqrt{R_{Px}^2 + R_{Py}^2} = \sqrt{\left(\sum X_k\right)^2 + \left(\sum Y_k\right)^2}. \tag{11}$$

---

* We consider a plane system of converging forces and so the plane in which the forces lie may be taken as the coordinate plane.

The angles between the resultant and the coordinate axes (and, hence, the direction of the resultant) are given by the formulas

$$\left.\begin{aligned}\cos(\widehat{\mathbf{R}_p, x}) &= \frac{R_{px}}{R_p} = \frac{\sum X_k}{\sqrt{(\sum X_k)^2 + (\sum Y_k)^2}},\\ \cos(\widehat{\mathbf{R}_p, y}) &= \frac{R_{py}}{R_p} = \frac{\sum Y_k}{\sqrt{(\sum X_k)^2 + (\sum Y_k)^2}}.\end{aligned}\right\} \qquad (12)$$

Formulas (11) and (12) make it possible to determine analytically the magnitude and direction of the resultant of a system of converging forces. The projections of the component forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$ are readily computed from formulas (6):

$$\begin{aligned}X_1 &= P_1 \cos(\widehat{\mathbf{P}_1, x}),\\ X_2 &= P_2 \cos(\widehat{\mathbf{P}_2, x}),\\ &\ldots\ldots\ldots\ldots\\ X_n &= P_n \cos(\widehat{\mathbf{P}_n, x}),\\ Y_1 &= P_1 \cos(\widehat{\mathbf{P}_1, y}),\\ Y_2 &= P_2 \cos(\widehat{\mathbf{P}_2, y}),\\ &\ldots\ldots\ldots\ldots\\ Y_n &= P_n \cos(\widehat{\mathbf{P}_n, y}).\end{aligned}$$

Problem 7. Given the forces $P_1 = 10$ N, $P_2 = 10$ N, $P_3 = 15$ N, $P_4 = 20$ N applied to one point. The directions of the forces are indicated in Fig. 29. The angles between the vectors are:

$$(\widehat{\mathbf{P}_1, \mathbf{P}_2}) = 50^\circ, \quad (\widehat{\mathbf{P}_2, \mathbf{P}_3}) = 70^\circ, \quad (\widehat{\mathbf{P}_3, \mathbf{P}_4}) = 80^\circ.$$

Determine analytically the resultant of the given forces.

Solution. We take the common point O of the application of the given forces as the origin of coordinates, and the line of action of the force $\mathbf{P}_1$ as the x-axis. The y-axis is then drawn perpendicularly to the x-axis, as shown in Fig. 29.

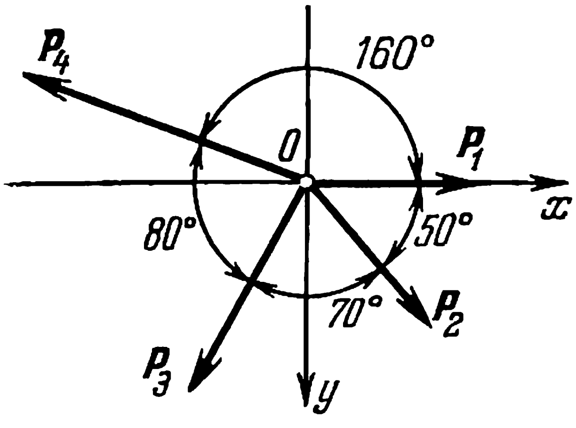


FIG. 29

Projecting all the forces on the coordinate axes, we obtain, by (6), the following:

$X_1 = P_1 \cos 0° = P_1 = 10$ N,

$X_2 = P_2 \cos 50° = 10 \times 0.643 = 6.43$ N,

$X_3 = -P_3 \cos (180° - 120°) = -P_3 \cos 60° = -15 \times 0.5 = -7.5$ N,

$X_4 = -P_4 \cos (180° - 160°) = -P_4 \cos 20° = -20 \times 0.940 = -18.8$ N,

$Y_1 = P_1 \cos 90° = 0$,

$Y_2 = P_2 \cos 40° = 10 \times 0.766 = 7.66$ N,

$Y_3 = P_3 \cos 30° = 15 \times 0.866 = 12.99$ N,

$Y_4 = -P_4 \cos 70° = -20 \times 0.342 = -6.84$ N.

From this, via (10), we determine the projections of the resultant:

$$R_{Px} = \sum X_k = -9.87 \text{ N}, \qquad R_{Py} = \sum Y_k = 13.81 \text{ N}.$$

Using formula (11), we determine the magnitude of the resultant:

$$R_P = \sqrt{R_{Px}^2 + R_{Py}^2} = \sqrt{(-9.87)^2 + 13.81^2} \approx 17 \text{ N}.$$

The direction of the resultant is determined from formula (12):

$$\cos (\widehat{\mathbf{R}_P, x}) = \frac{R_{Px}}{R} = -\frac{9.87}{17} = -0.580,$$

$$\cos (\widehat{\mathbf{R}_P, y}) = \frac{R_{Py}}{R} = \frac{13.81}{17} = 0.812.$$

From this we find (using tables or a slide rule) the angles between the line of action of the resultant $\mathbf{R}_P$ and the positive direction of the coordinate axes:

$$(\widehat{\mathbf{R}_P, x}) = 125^\circ 30', \qquad (\widehat{\mathbf{R}_P, y}) = 35^\circ 30'.$$

Sec. 14. Conditions of equilibrium of a plane system of converging forces

Any system of converging forces may be replaced by a resultant. Clearly, if such a system of converging forces is in equilibrium, that is, is equivalent to zero, then the resultant must be zero.

That the resultant be equal to zero is a necessary and sufficient condition for the equilibrium of a system of converging forces.

In accordance with the two methods of determining a resultant, the condition of equilibrium of a plane system of converging forces may be expressed in two forms:

1. *The condition of equilibrium in geometric form.* Geometrically, the resultant of converging forces is given as the closing side of a force polygon. If the resultant is zero, then it is necessary that the closing side be zero as well and, hence, that the force polygon close by itself. We thus get the following condition: *for the equilibrium of a system of converging forces, it is necessary and sufficient that the force polygon constructed for the system be closed.*

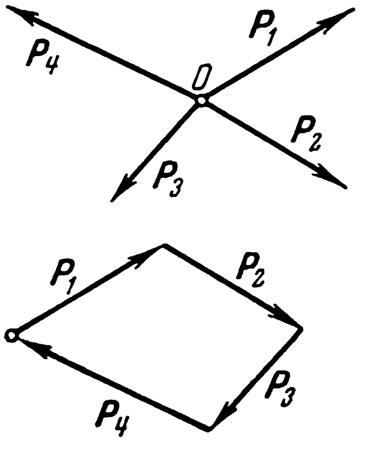


FIG. 30

Figure 30 shows a closed force polygon constructed for a plane system of converging forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ in equilibrium. Note that in a closed force polygon, the terminus of the vector of the last force coincides with the origin of the vector of the first force, and the arrows of the vectors of all forces point in the same direction of traversal of the perimeter of the polygon.

2. *The condition of equilibrium in analytic form.* Analytically the magnitude of the resultant is given by formula (11):

$$R_P = \sqrt{(\sum X_k)^2 + (\sum Y_k)^2}$$

But if $R_P = 0$, then so also is the radicand equal to zero. Since the terms under the radical sign, as the squares of certain numbers (it is immaterial whether they are positive or negative), are always positive, it follows that R_P can be zero only if each of the summands is equal to zero separately:

$$\sum X_k = 0 \quad \text{and} \quad \sum Y_k = 0. \tag{13}$$

These equations, called the equations of equilibrium, express in analytic form the necessary and sufficient conditions of the equilibrium of the forces.

For the equilibrium of a plane system of converging forces, it is necessary and sufficient that the sums of the projections of all forces on each of any two mutually perpendicular axes lying in the plane of action of the forces be separately equal to zero.

Sec. 15. A theorem on the equilibrium of three nonparallel forces lying in one plane

If three nonparallel forces lying in one plane balance, the lines of their action intersect in one point.

Proof. Suppose three nonparallel mutually balancing forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ located in one plane (Fig. 31) are applied at three points A_1, A_2 and A_3. Since the forces are not parallel, the lines of action of two of them, say $\mathbf{P}_1$ and $\mathbf{P}_2$, must intersect at some point A. Translating the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ along the lines of action to the point A and combining them by the parallelogram rule, we obtain a resultant force $\mathbf{R}_P$. We may now assume that only two forces are acting on the body: $\mathbf{R}_P$ and $\mathbf{P}_3$. These forces must be equal in magnitude and opposite in direction along a single straight line because

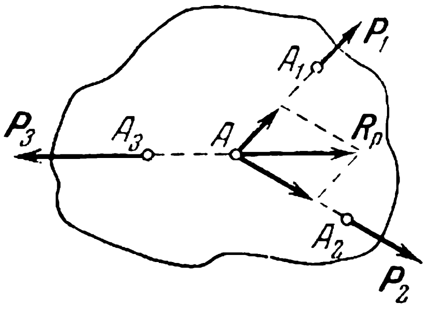


FIG. 31

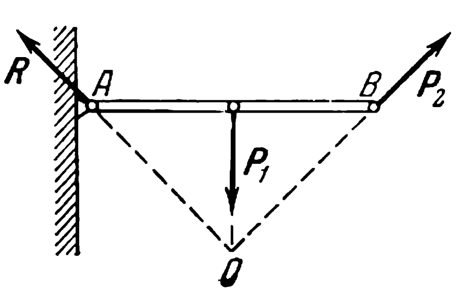


FIG. 32

it is stated that the given forces must balance. Therefore the line of action of force $\mathbf{P}_3$ must coincide with the line of action of the force $\mathbf{R}_p$ and, hence, must pass through A, at which point the lines of action of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ intersect.

This condition for the equilibrium of three nonparallel forces is a necessary condition but it is not sufficient. We can assert that if three nonparallel forces are in equilibrium, then the lines of their action intersect in a single point. The converse is not true however: if the lines of action of three forces intersect in a single point, it does not at all follow therefrom that these three forces are in equilibrium.

If a body is in equilibrium under the action of three nonparallel forces (these include any unknown reaction constraints) lying in one plane, the direction of the constraint reactions may be determined from the preceding theorem, according to which the lines of action of the forces must intersect in a single point.

In Fig. 32 we have a method, based on the given theorem, for determining the direction of the constraint reaction (the reaction **R** of the fixed hinge A). As is evident from the drawing, to determine the direction of the reaction it is necessary to continue the lines of action of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied to the body (the rod AB) up to their intersection at point O. The straight line joining O and the fixed point A (the fixed bolt of the hinge is regarded as a point) is the line of action of the reaction **R** of the hinge A.

Sec. 16. Remarks on problem solving that involves the equilibrium of a system of forces

When solving problems on equilibrium, and not only those involving converging forces but also forces located arbitrarily, it is advisable to adhere to the following procedure. First of all, it is essential to have a clear picture of the conditions of the problem, what precisely is required, so that all subsequent operations are of a quite definite purpose. Then the body or point whose equilibrium is under discussion in the problem must be relieved of its constraints (they are removed by the principle of removability, Sec. 7, and replaced by appropriate reactions). A drawing should be made to some kind of scale with the active forces and all constraint reactions indicated that are applied to the body. The constraint reactions are almost always unknown. Determining their magnitude and sometimes the magnitude and direction on the basis of given known forces applied to the body is exactly what constitutes the contents of most problems in statics. In order to determine the directions of the constraint reactions, make use of the reasoning given earlier in Sec. 7.

After all the active forces and constraint reactions applied to the given body in equilibrium have been established, we make use of the conditions of equilibrium of those forces in geometric or analytic form, depending on which form proves to be simpler and more convenient in the given problem. In the former case, for a system of converging forces we determine the sought-for forces or other unknowns in the problem by constructing a closed force polygon or in a purely graphical manner by constructing the force polygon to a definite scale, or by computing its sides and angles via the rules of geometry and trigonometry. In the latter case (analytical method), we find the desired quantities by using the method of projections from the equilibrium equations (13):

$$\sum X_k = 0, \quad \sum Y_k = 0.$$

In this method, it is convenient to take for the origin of coordinates that point at which the forces converge, and then place the coordinate axes so that the projections of the forces on these axes are found in the simplest possible fashion. It often turns out to be convenient to place one of the axes perpendicular to the line of action of one of the unknown forces. The projection of that unknown force is then eliminated from the corresponding equilibrium equation and the solution of the equilibrium equation is simplified.

At first, prior to acquiring skills in setting up equilibrium equations, it is useful to tabulate the values of the projections of the forces on the coordinate axes. This simplifies checking the solution and seeking out possible errors.

Problem 8. A lantern weighing $G = 20$ N is suspended between two walls (Fig. 33*a*). The left-hand rope forms an angle of $\alpha = 45°$ with the wall, the right-hand rope forms an angle $\beta = 30°$. Find the tension experienced by both ropes.

Solution. It is required to determine the tension of the ropes. The ropes are stretched by the lantern. We know the weight of the lantern.

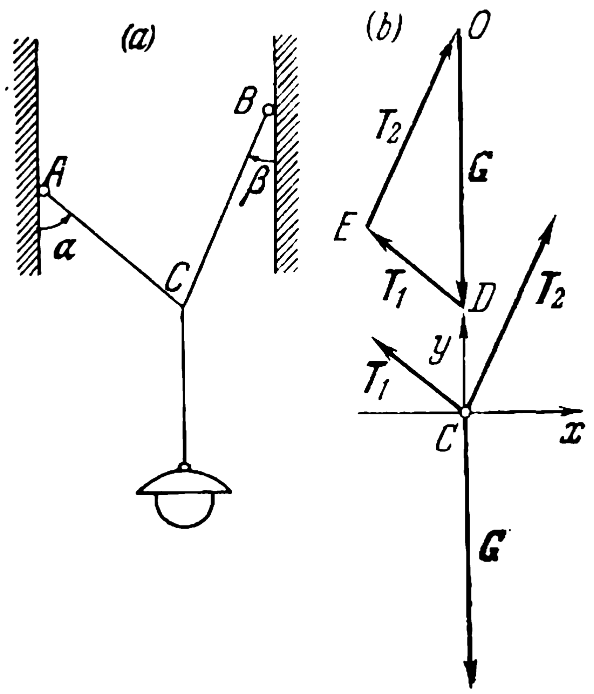


FIG. 33

The active force G, the weight (gravity) of the lantern, acts on point *C*. That point is not free, the constraints are accomplished by the ropes *CA* and *CB*. Consider the equilibrium of the point *C*. Let us remove the constraints from that point (by cutting the ropes mentally) and replace their action by reactions. Then *C* may be regarded as free and as being in equilibrium under the action of three forces: the active force G and the reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ of the ropes (Fig. 33*b*). These reactions are numerically equal to the desired tensions of the ropes. Instead of determining the tensions of the ropes, we determine their reactions. Let us consider several methods of solving this problem.

(a) *Graphical method.* Choose a definite scale, say 1 N per mm, lay off to that scale, from an arbitrary point *O* in the direction of the force G, a vector of that force of length 20 mm (Fig. 33*b*), and then from the origin *O* of the vector draw a straight line *OE* parallel to the rope *BC*, and from the terminus *D* a straight line *DE* parallel to the rope *CA*. Since all the forces applied to *C* are balanced, the resulting force triangle *ODE* must be closed, that is, all arrows in the triangle must traverse the triangle in one direction. The sides *DE* and *EO* of the triangle yield the magnitudes and directions of the desired reactions of the ropes. To find their magnitudes and, hence, the tensions of the ropes, it remains to measure (in the chosen scale) the lengths of the sides *DE* and *EO* of the triangle.

(b) *Geometrical method.* The desired sides of the force triangle may be found not only by direct measurement but also by computation.

From the method of constructing the force triangle *ODE* it is clear that $\angle ODE = \alpha$ and $\angle EOD = \beta$ (Figs. 33*a*, 33*b*). By the law of sines, we have

$$\frac{T_1}{\sin\beta} = \frac{T_2}{\sin\alpha} = \frac{G}{\sin[180^\circ - (\alpha+\beta)]} = \frac{G}{\sin(\alpha+\beta)}$$

whence

$$T_1 = \frac{G\sin\beta}{\sin(\alpha+\beta)} = \frac{20\sin 30^\circ}{\sin(45^\circ+30^\circ)} = \frac{20\times 0.5}{0.966} \approx 10.4 \text{ N},$$

$$T_2 = \frac{G\sin\alpha}{\sin(\alpha+\beta)} = \frac{20\sin 45^\circ}{\sin 75^\circ} = \frac{20\times 0.707}{0.966} \approx 14.7 \text{ N}.$$

(c) *Analytical method.* Take the origin of coordinates at the point *C* whose equilibrium we are considering. Send the x-axis horizontally rightwards, the y-axis vertically upwards (Fig. 33*b*). From a comparison of Figs. 33*a* and 33*b* it is evident that the angles $\widehat{(\mathbf{T}_1, y)} = \alpha = 45^\circ$ and $\widehat{(\mathbf{T}_2, y)} = \beta = 30^\circ$. Using formula (6), we find the values of the projections of all forces on the chosen coordinate axes and tabulate them for convenience.

Force	Projections of force on axes	
	x	y
$\mathbf{G}$	0	$-G$
$\mathbf{T}_1$	$-T_1\cos(90^\circ-\alpha) = -T_1\cos 45^\circ$	$T_1\cos 45^\circ$
$\mathbf{T}_2$	$T_2\cos(90^\circ-\beta) = T_2\cos 60^\circ$	$T_2\cos 30^\circ$

Here the equilibrium equations (13) take the form

$$\sum X_k = -T_1\cos 45^\circ + T_2\cos 60^\circ = 0,$$

$$\sum Y_k = -G + T_1\cos 45^\circ + T_2\cos 30^\circ = 0,$$

$$\cos 45^\circ = \frac{\sqrt{2}}{2} = 0.707, \quad \cos 60^\circ = 0.5,$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = 0.866 \text{ and } G = 20\text{ N}.$$

Substituting the values found, we get

$$-0.707\,T_1 + 0.5\,T_2 = 0,$$

$$-2 + 0.707\,T_1 + 0.866\,T_2 = 0.$$

Solving this system, we find

$$T_2 = \frac{20}{1.366} = 14.6\ \mathrm{N}, \quad T_1 = \frac{0.5 \times 14.6}{0.707} = 10.3\ \mathrm{N}.$$

We thus see that the problem may be solved in different ways*. The choice of procedure depends on the nature of the problem and on the requirements of accuracy. The geometrical method is inconvenient if the number of forces exceeds three.

Problem 9. Rods *AB* and *BC* (Fig. 34*a*) are joined together and to a vertical wall by means of hinges. The rods are of length: $AB = a$ and $BC = b$. The distance $AC = c$. Determine the reactions of the rods *AB* and *BC* on the link bolt *B* if a load weighing *G* is suspended from it.

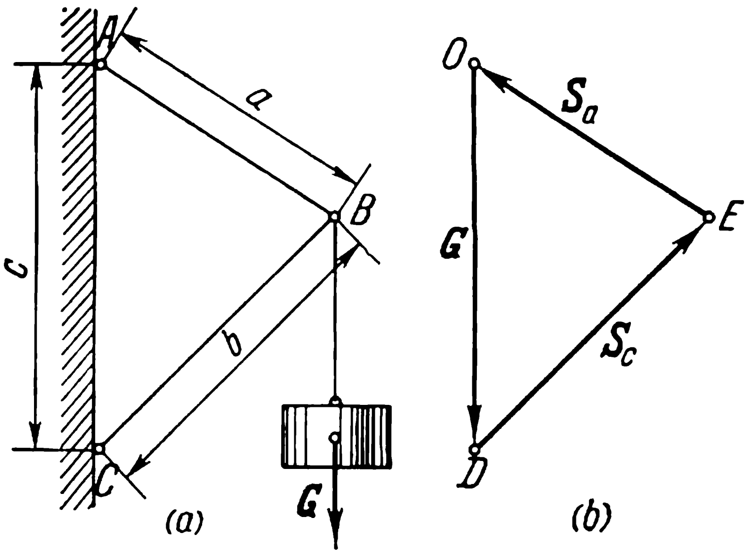


FIG. 34

Solution. The link bolt *B* is acted on by the following forces: the vertical force of gravity **G** and the reactions $\mathbf{S}_a$ and $\mathbf{S}_c$ of the rods *AB* and *BC* directed along the rods. Since the point *B* is in equilibrium under the action of the system of converging forces applied to it (**G**, $\mathbf{S}_a$ and $\mathbf{S}_c$), the force polygon for this system must be closed. We construct that polygon (Fig. 34*b*) by laying off (to some scale) from an arbitrary point *O* the vector $\overrightarrow{OD} = \mathbf{G}$ and drawing from its end-points the straight lines *OE* and *DE* that are parallel to the desired reactions, that is, parallel to the rods *AB* and *BC*. In the chosen scale, the lengths of the sides *OE* and *DE* of the resulting force triangle yield the desired magnitudes of the reactions $\mathbf{S}_a$ and $\mathbf{S}_c$. To calculate them, the easiest thing is to take advantage of the proportionality of the sides of the triangles *ODE* and *ACB* (the triangles being similar): $S_a/a = S_c/b = G/c$, whence

$$S_a = G\frac{a}{c} \quad \text{and} \quad S_c = G\frac{b}{c}.$$

---

* Besides the methods discussed above, we could also approach the problem of finding the tension of the ropes by resolving the force **G** into components along the specified directions of the ropes.

Problem 10. A load of weight $G = 20$ kN is hoisted by a winch (Fig. 35) with the aid of a rope thrown over a fixed block at point B. Ignoring friction in the block and also the dimensions, determine the stresses in the bars AB and BC. The angles are indicated in the drawing.

Solution. Consider the equilibrium of point B of the axis of a fixed block. Replace the constraints imposed on it by appropriate reactions. Since the bars AB and BC are loaded in the joint and the bars are joined by hinges, the rods can only undergo tension or compression and, hence, the reactions of the bars will be directed along their axes. The bar AB is obviously under tension and its reaction $\mathbf{S}_1$ is directed from B to A, the bar BC is under compression and its reaction $\mathbf{S}_2$ is directed from C to B. The magnitude T of the reaction of the rope is equal to its tension, that is, equal to the weight G of the load because, in the absence of friction, both parts of the rope must be under the same tension.

To summarize, then, at the point B, which is in equilibrium, are applied four forces $\mathbf{G}$, $\mathbf{T}$, $\mathbf{S}_1$, and $\mathbf{S}_2$, and the magnitudes of the last two forces are unknown. It is required to calculate them.

The simplest way to do this is by the method of projections. Take point B for the coordinate origin, the line of action of one of the applied forces $\mathbf{S}_1$, for the direction of the x-axis, and draw the y-axis perpendicular to it, as shown in Fig. 35.

Project all forces applied to point B on the chosen coordinate axes and draw up the following table.

Force	Projections of force on axes		Force	Projections of force on axes	
	x	y		x	y
G	0	G	$\mathbf{S}_1$	$-S_1$	0
T	$-T\cos 60°$	$T\cos 30°$	$\mathbf{S}_2$	$S_2\cos 30°$	$-S_2\cos 60°$

Solving the system of equations of equilibrium,

$$\sum X_k = -T_2\cos 60° - T_1 + S\cos 30° = 0,$$

$$\sum Y_k = G + T_2\cos 30° - S\cos 60° = 0,$$

we find $S_2 = 74.6$ kN, $S_1 = 54.6$ kN.

Problem 11. A rod AB (Fig. 36a) is attached to a vertical wall by means of a hinge A and is held at an angle of 60° to the wall by a rope BC that forms an angle of 60° with the rod. Determine the magnitude and the direction of the reaction of the hinge if it is known that the weight of the rod $G = 120$ N is applied to its midpoint.

Solution. Considering the equilibrium of the rod AB, replace the constraints imposed on it (the rope BC and the hinge A) by their reactions. The reaction of the rope **T** is directed along the rope from *B* to *C*.

The direction of the reaction of the fixed hinge *A*, as we know from Sec. 7.2, is generally speaking indeterminate. But since the rod under the action of three nonparallel forces applied to it is in equilibrium, the lines of action of the three forces must intersect at one point.

Continue the line of action of the force of gravity (weight of the rod) up to the intersection, at point *D*, with the line of action of the reaction **T** of the taut rope. Joining *D* and *A*, we obtain the line of action of the reaction **R** of the hinge *A*. Knowing the magnitude and the direction of the force **G** that belongs to a system of three forces, which converge at the point *D* and are in a state of equilibrium, and also knowing the directions of two other forces, **R** and **T**, it is not difficult to find the magnitudes of the latter by constructing a force triangle.

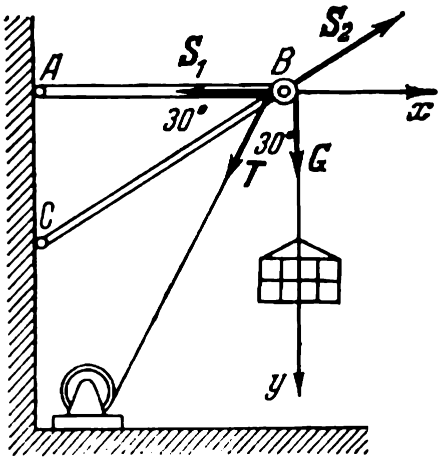
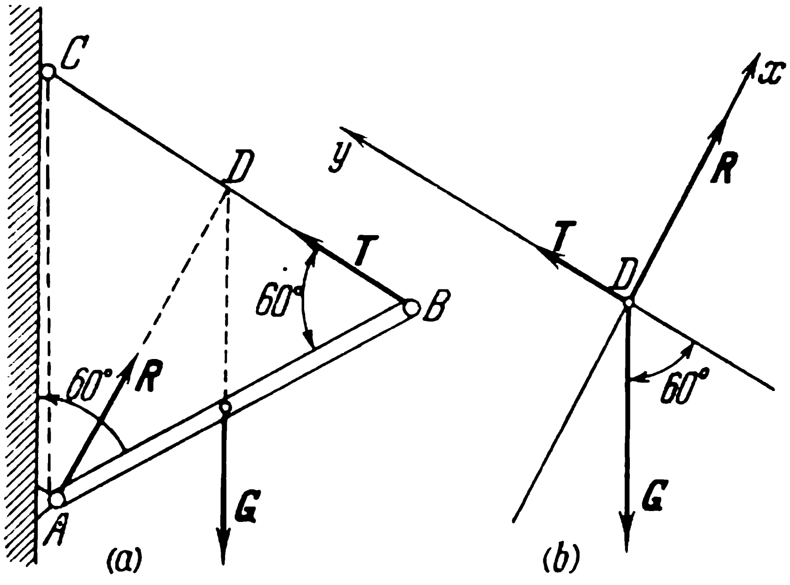


FIG. 35



(a) (b)

FIG. 36

Let us solve the problem analytically. The triangle *ABC*, as is evident from the drawing, is an equilateral triangle.

Furthermore, since the line of action of the force of gravity **G** is parallel to the straight line *AC* and bisects the side *AB* of the triangle *ABC*, it also bisects, at point *D*, the other side, *BC*, of that triangle. Consequently, the line of action of the reaction **R** passing through *D* is a median in the triangle *ABC*; at the same time, by the property of any isosceles triangle, it is the altitude of the triangle and the bisector of the angle *BAC*. From this we find that $\angle CAD = \angle DAB = 60°/2 = 30°$ and $\angle ADC = (\widehat{\mathbf{R}, \mathbf{T}}) = 90°$. To determine the magnitude of the hinge reaction, take point *D* as the coordinate origin and draw coordinate axes as shown in Fig. 36*b*. By projecting all the forces convergent at *D* on the x-axis and setting up the appropriate equation of equilibrium, we obtain $\sum X_k = R - G\cos 30° = 0$, whence $R = G\cos 30° = 120 \times 0.866 \approx 104$ N.

By projecting the forces on the y-axis, one can also easily find the tension **T** of the rope.

Sec. *17. Truss analysis. Essentials. The method of isolating joints

A truss is a fixed lattice structure consisting of bars joined at the end points.

These structural members which appear at first glance to be fragile take the place of solid structural members, thus considerably reducing the overall weight of the structure with a concomitant saving of materials. They find extensive and diverse use in bridge and roof construction, in hoisting cranes, towers, aircraft, and so forth.

If the axes of all rods of the truss and applied forces lie in a single plane, the truss is called a simple *plane* truss. The points of intersection of the truss are termed *joints.*

If the bars of the truss were weightless and were joined at their end points by ideal hinges, then, as we know (see Sec. 7.6), given loads applied to the joints, the reactions of the bars would always be directed along their axes and the bars could only be under tension or compression. Now, as we know from the strength of materials course, when the bar material is under tension or compression, it is utilized to the greatest possible extent*. In reality, of course, bars have weight and are not hinge-connected but are welded or riveted in place. This gives rise to additional stresses in the bars. But since the stresses are small, they are disregarded in order to simplify analysis; the truss is regarded as composed of straight bars joined together at their ends by ideal hinges and, hence, experiencing only tension and compression. The resulting inaccuracy in the analysis is quite permissible for all practical purposes. Here, all forces (loads) acting on the truss must be applied at the joints.

But in actuality, loads often appear that are distributed along the bars of the truss (wind pressure, the weight of snow, etc.). Ordinarily, such forces are small compared with the loads applied at the joints and so are often disregarded; if not, they have to be decomposed into components applied at the joints.

---

* This explains why one always strives to design machine parts and structures so that they experience tension and compression and not bending.

For a truss to be useful as a structural member, it must be a stable (rigid) system.

If we join three bars (*1*, *2*, and *3* in Fig. 37), we obtain the simplest possible kind of rigid system: the bars do not allow joints *I*, *II* and *III* to change their relative positions.

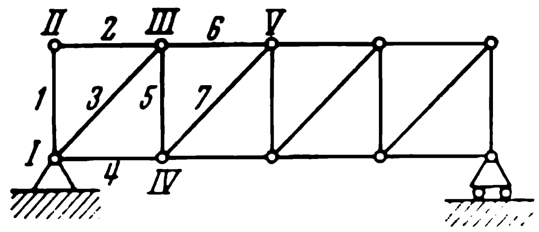


FIG. 37

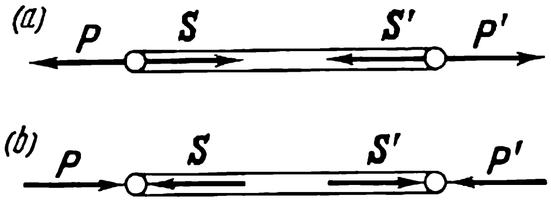


FIG. 38

To add another joint, *IV*, and retain rigidity, requires only two more bars, *4* and *5*. To adjoin yet another joint *V* requires only two more bars, *6* and *7*, and so on. Thus, if a truss has n joints, then to obtain the first three requires three bars, to obtain the remaining $(n - 3)$ joints requires $2(n - 3)$ bars. Consequently, the total number of bars necessary for forming a truss with n joints is

$$k = 3 + 2(n - 3)$$

or

$$k = 2n - 3. \tag{14}$$

This is the condition of rigidity of a plane system of bars with hinged end points. The system will not be rigid if the number of bars is less. Given a larger number of bars in the truss, the system will be *statically indeterminate*, which means that methods of statics* no longer suffice for its analysis.

To analyze a truss means to determine the reactions of the supports of the truss and the forces (acting in the bars) that appear due to loads applied to the truss. In the design of a truss, these forces must be known in order to choose bars of appropriate strength.

---

* Statically indeterminate problems are discussed briefly in Sec. *34.

For the whole structure, regarded as an absolutely rigid body, the load on the truss and the reactions of its supports are external forces, and the forces in its bars are internal forces. These forces are numerically equal to the reactions of the bars on the joints of the truss, and for that reason determining the forces in the bars amounts to determining the reactions of those bars. Now to determine the nature of the forces, note the following.

If (see Fig. 38a, p. 73) a bar undergoes tension due to forces **P** and **P′** acting on the bar, its reactions **S** and **S′** acting on opposite joints of the truss are directed along the bar from the joints and towards each other. Now if the rod is compressed (Fig. 38*b*), its reactions will be directed along the bar away from each other towards the joints. Hence, if we know the directions of the reactions of the bars, we can always determine whether it is undergoing tension or compression. *If the reaction of the bar on a given joint is away from the joint into the bar, the bar undergoes tension*; *if the reaction of the bar on the joint is directed outwards, the bar undergoes compression.*

It is also clear (Fig. 38) that the reactions of each bar acting on the joints it connects are equal in magnitude and opposite in direction.

Truss analysis usually begins with a determination (graphical or analytical) of the reactions of its supports. How that is done in the general case will be explained later on in the course of statics.

To determine the forces in the bars of a truss, we have at our disposal a number of methods, both graphical and analytical. One such method, called the *method of cutting out joints*, has the following underlying considerations.

If the truss as a whole is in a state of equilibrium, then each part of it, each joint, is in equilibrium as well. If we mentally isolate any joint of the truss, we thus relieve that joint of the constraints (of the bars joining it to the remaining part of the truss) and we have to replace those constraints with reactions of the appropriate bars on that joint. Then, by applying the conditions of equilibrium to the system of forces converging at one or another joint, we are able to determine the reactions of the bars of the given joint.

This problem may be solved analytically (by setting up the equations of equilibrium for forces applied to the other joints of the truss) or graphically (by constructing a closed force polygon).

Any solution of an equilibrium problem of a plane system of converging forces is possible only if the number of unknowns does not exceed two. For this reason, we have to start isolate ("cutting out") the joint that has not more than two converging bars (ordinarily, those are bars on the supports), and only then go on to other joints, paying particular attention to the joint not having more than two bars with unknown reactions.

Problem 12. Determine the reactions of the supports and the forces in the bars of a truss as depicted to an arbitrary scale in Fig. 39*a*. The left-hand support is connected with the truss by a cylindrical hinge *A*, and the right-hand support *B* rests on rollers that can move along an inclined plane at 45° to the horizon. A vertical force $P = 60$ kN is applied at joint *E* of the truss. All bars are of the same length. The weight of the bars may be ignored.

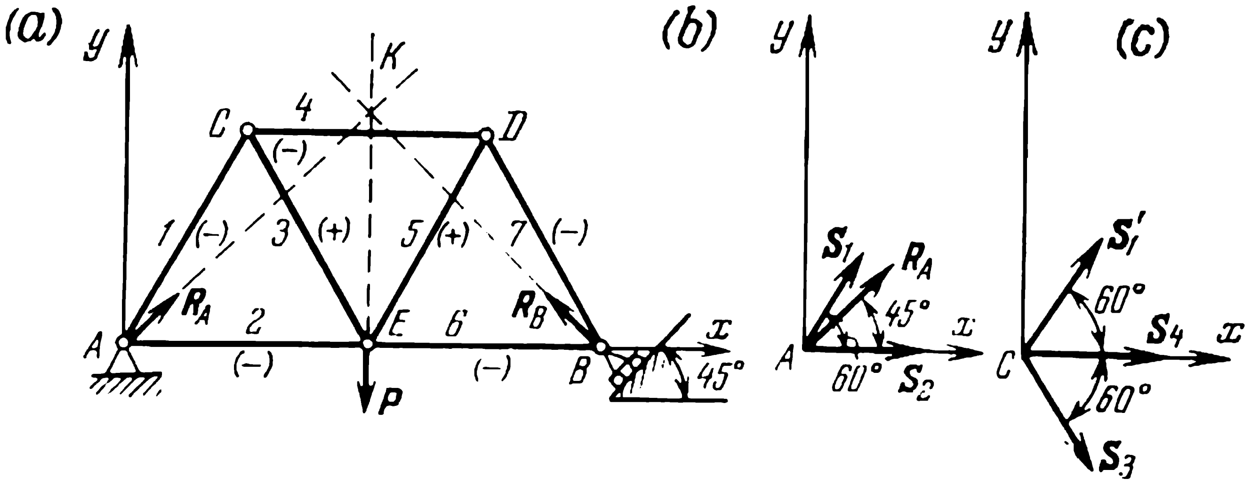


FIG. 39

Solution. One active force **P** is applied to the truss and there are two constraints imposed (the supports *B* and *A*). We get rid of the constraints by replacing them with reactions. The support *B* rests on rollers, and so its reaction $\mathbf{R}_B$ is (in accordance with what was said in Sec. 7.3) perpendicular to the support plane. The direction of the reaction $\mathbf{R}_A$ of a fixed cylindrical hinge is not known beforehand. Prolong the lines of action of the forces **P** and $\mathbf{R}_B$ to their intersection at some point *K* (Fig. 39*a*). By the theorem on the equilibrium of three nonparallel forces, the line of action of the reaction $\mathbf{R}_A$ at point *A* must also pass through the common point *K*.

Number all the bars as shown in Fig. 39*a*. Let us agree to denote the reactions of the bars by S and subscripts indicating the number of each bar.

What follows is an analytical solution of the problem.

The lines of action of the forces $\mathbf{R}_A$ and $\mathbf{R}_B$ pass through points A and B, which are equidistant from the line of action of the force **P**, and intersect at the point K lying on that line. Hence, the line of action of the force $\mathbf{R}_A$ forms the same angle with the horizontal as does the line of action of the force $\mathbf{R}_B$, an angle of 45°.

Now set up the equations of equilibrium of the truss as a whole under the action of external forces **P**, $\mathbf{R}_A$ and $\mathbf{R}_B$ applied to it. For the projection axes we take the axes shown in Fig. 39*a*:

$$\sum X_k = R_A \cos 45^\circ - R_B \cos 45^\circ = 0,$$

$$\sum Y_k = R_A \cos 45^\circ - P + R_B \cos 45^\circ = 0.$$

Solving the given system of equations and substituting the value of the force $P = 60$ kN, we find $R_A = R_B = 42.4$ kN.

To determine the forces in the bars, isolate first the joint A (Fig. 39*a*) that has only two bars *1* and *2* with unknown reactions S_1 and S_2 and the known reaction $\mathbf{R}_A$ of the support A. For the time being, we assume that the bars are under tension and so their reactions on the joint A are directed away from the joint into the appropriate bar (Fig. 39*b*).

Setting up and solving the system of equilibrium equations for forces applied to joint A, we get

$$\sum X_k = S_1 \cos 60^\circ + R_A \cos 45^\circ + S_2 = 0,$$

$$\sum Y_k = S_1 \cos 30^\circ + R_A \cos 45^\circ = 0,$$

whence $S_1 = -34.2$ kN, $S_2 = -12.6$ kN.

The negative values obtained for the reactions S_1 and S_2 show that in reality the directions of the reactions of bars *1* and *2* are opposite to the directions we took when setting up the equilibrium equations and, hence, the bars *1* and *2* are not under tension but under compression.

We can now examine the next joint, but it must not have more than two bars with unknown reactions. C (Fig. 39*a*) is just such a joint. Three forces converge in that joint: the reaction S_1' of bar *1* that has already been found in magnitude, and the unknown reactions S_3 and S_4 of bars *3* and *4*. We found out that bar *1* is under compression; hence, the reactions of that bar on the joints must be directed outwards from the joints. Consequently, the reaction S_1' of bar *1* on joint C must be in the direction indicated in Fig. 39*c*. For the time being, we again regard bars *3* and *4* as under tension and therefore their reactions on joint C as directed from that joint into the appropriate bars.

The equilibrium equations for forces applied to joint *C* (Fig. 39*c*) are of the form

$$\sum X_k = S_1' \cos 60^\circ + S_4 + S_4 \cos 60^\circ = 0,$$

$$\sum Y_k = S_1' \cos 30^\circ - S_3 \cos 30^\circ = 0.$$

Solving the given system of equations and substituting the values $S_1' = -S_1 = 34.2$ kN, we find $S_3 = S_1' = 34.2$ kN and $S_4 = -34.2$ kN.

The negative value obtained for the reaction S_4 of bar *4* shows that the direction of that reaction is opposite the preliminarily assumed one and, hence, that bar is under compression. The positive value obtained for the reaction S_3 of bar *3* confirms the fact that that bar is under tension.

After finding the values of the reactions of bars *2* and *3*, we could pass on to a consideration of the equilibrium of joint *E*, to which are applied two unknown forces (the forces of reaction of bars *5* and *6*) in addition to the reaction forces of bars *2* and *3* and the force **P**. And then pass on to considering the equilibrium of joint *D* to which are applied the now known reactions of bars *4* and *5* and the unknown reaction of bar *7*. But this need not be done in the given problem. Due to the symmetric nature of the truss and the symmetric arrangement of the external forces applied to it—**P**, $\mathbf{R}_A$ and $\mathbf{R}_B$—it is obvious that $S_5 = S_3 = 34.2$ kN, $S_6 = S_2 = -12.6$ kN and $S_7 = S_1 = -34.2$ kN.

In Fig. 39*a*, the bars under tension are marked with a plus sign (+), those under compression with a minus sign (—).

CHAPTER III

SYSTEMS OF TWO PARALLEL FORCES

Sec. 18. The composition of two parallel forces acting in the same direction

The parallelogram rule for the composition of parallel forces is inapplicable directly since the point of intersection of parallel forces lies at infinity.

In order to derive the rule for the composition of two parallel forces, we replace the forces by an equivalent system of two converging forces.

We first consider a system of two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in the same direction (Fig. 40). At the points of application of the forces*, A and B, we apply two forces $\mathbf{T}_1$ and $\mathbf{T}_2$ equal in magnitude and acting in opposite directions along the straight line AB. Now use the parallelogram rule to combine force $\mathbf{P}_1$ with $\mathbf{T}_1$ and force $\mathbf{P}_2$ with $\mathbf{T}_2$; we obtain two converging forces $\mathbf{R}_1$ and $\mathbf{R}_2$. Now translate the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ along the lines of action to the point O of intersection of the lines. To determine the magnitude and

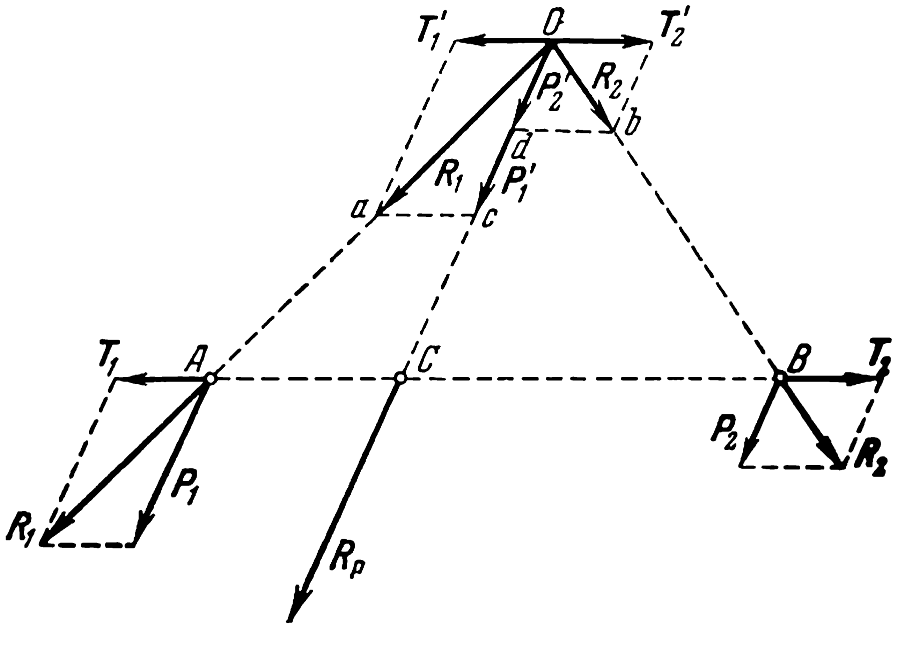


FIG. 40

---

* For the points of application of the forces we can take any points on their lines of action.

the line of action of the resultant of the given forces, we now perform the inverse operations; we decompose the force $\mathbf{R}_1$ into two components: $\mathbf{P}_1'$ and $\mathbf{T}_1'$ which are parallel to the forces $\mathbf{P}_1$ and $\mathbf{T}_1$, and we decompose $\mathbf{R}_2$ into two components $\mathbf{P}_2'$ and $\mathbf{T}_2'$ parallel to $\mathbf{P}_2$ and $\mathbf{T}_2$. From the pairwise equality of the parallelograms constructed at the points A, B and O, it follows that the resulting components $\mathbf{P}_1'$, $\mathbf{T}_1'$, $\mathbf{P}_2'$ and $\mathbf{T}_2'$ are correspondingly equal in magnitude to the forces $\mathbf{P}_1$, $\mathbf{T}_1$, $\mathbf{P}_2$ and $\mathbf{T}_2$. Thus the system of two parallel forces is reduced to a system of four forces applied to the single point O.

The forces $\mathbf{T}_1'$ and $\mathbf{T}_2'$ are equal in magnitude and opposite in direction (acting along the same line) and so may be discarded since they cancel. There remain two forces $\mathbf{P}_1'$ and $\mathbf{P}_2'$ that act along the same straight line and in the same direction. Their resultant $\mathbf{R}_P$ is directed along the same straight line parallel to the lines of action of the given forces, in the same direction, and equal in magnitude to their sum:

$$R_P = P_1 + P_2. \tag{15}$$

Now let us find the position of the line of action of the resultant. To do this we determine the position of point C, the point of intersection of that line with the straight line AB.

From the similarity of triangles OAC and Oac follows $AC/OC = ac/Oc$ or, taking into account the proportionality of the sides of the force triangle to the magnitudes of the corresponding forces, we obtain

$$\frac{AC}{OC} = \frac{T_1}{P_1}.$$

From the similarity of the triangles OCB and Odb it follows that

$$\frac{CB}{OC} = \frac{db}{Od} \quad \text{or} \quad \frac{CB}{OC} = \frac{T_2}{P_2}.$$

Dividing the first proportion by the second and taking into account that $T_1 = T_2$, we get

$$\frac{AC}{CB} = \frac{P_2}{P_1}. \tag{16}$$

Formula (16) may also be given in the form

$$\frac{P_2}{AC}=\frac{P_1}{CB}=\frac{P_1+P_2}{AC+CB}=\frac{R_P}{AB}. \qquad (16a)$$

Thus, the point C divides the straight line AB into parts that are inversely proportional to the component forces, and we thus arrive at the following conclusion: *the resultant of two parallel forces acting in the same direction is parallel to those forces, acts in the same direction and is equal in magnitude to their sum; the line of action of the resultant lies between the lines of action of the component forces at distances inversely proportional to the magnitudes of the forces.*

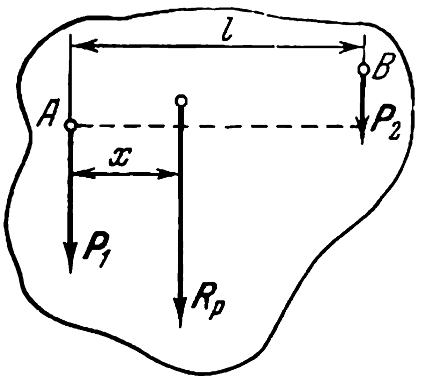


FIG. 41

Problem 13. Applied to a body (see Fig. 41) at the points A and B are two parallel forces acting in the same direction: $P_1 = 50$ N and $P_2 = 30$ N. Determine the magnitude and the line of action of the resultant if the distance between the lines of action of the given forces is $l = 1.6$ m.

Solution. The magnitude of the resultant $R_P = P_1 + P_2 = 80$ N. The distance of the line of action of the resultant $\mathbf{R}_P$ from the line of action of the force $\mathbf{P}_1$ is denoted by x. Using formula (16a), $\frac{P_2}{x}=\frac{R_P}{l}$, whence we find $x=\frac{P_2 l}{R_P}=\frac{30\times 1.6}{80}=0.6$ m.

Sec. 19. The composition of two forces unequal in magnitude and acting in opposite directions

Suppose we have two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ (Fig. 42) unequal in magnitude* and acting in opposite directions; $P_1 > P_2$. We continue the transformation in a manner similar to that done above. We apply to the points of application of the given forces A and B two forces $\mathbf{T}_1$ and $\mathbf{T}_2$ equal in magnitude and opposite in direction along the straight line AB. Add $\mathbf{P}_1$ and $\mathbf{T}_1$, $\mathbf{P}_2$ and $\mathbf{T}_2$ and translate the resultant forces $\mathbf{R}_1$ and $\mathbf{R}_2$ along their lines of action to the point O, the point of intersection of the lines.

---

* The case of equal (in magnitude) parallel forces acting in opposite directions will be considered in the next chapter.

Now decompose the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ into the components $\mathbf{P}'_1$, $\mathbf{T}'_1$, $\mathbf{P}'_2$, and $\mathbf{T}'_2$, which are respectively equal in magnitude and direction to the forces $\mathbf{P}_1$, $\mathbf{T}_1$, $\mathbf{P}_2$ and $\mathbf{T}_2$. Since the forces $\mathbf{T}'_1$ and $\mathbf{T}'_2$ balance, we obtain two forces $\mathbf{P}'_1$ and $\mathbf{P}'_2$ acting in opposite directions along a single straight line.

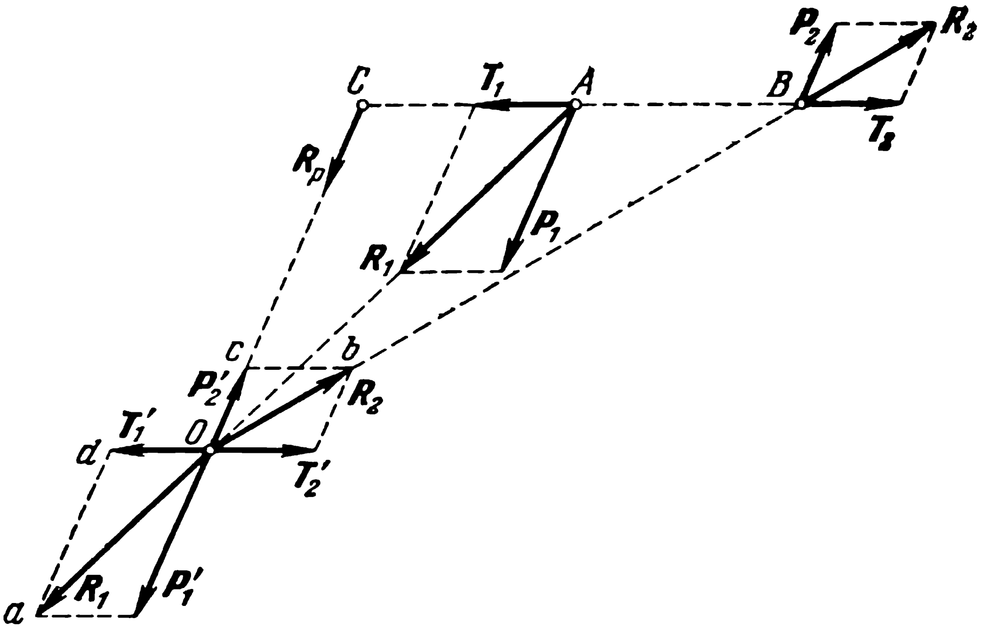


FIG. 42

Their resultant $\mathbf{R}_P$ acts along the same straight line parallel to the lines of action of the given forces $\mathbf{P}_1$ and $\mathbf{P}_2$, in the direction of the greater force, and is equal in magnitude to the difference between the forces:

$$R_P = P_1 - P_2. \tag{17}$$

The position of C, the point of intersection of the lines of action of the resultant $\mathbf{R}_P$ and the straight line AB, is determined from the similarity of the corresponding triangles.

From the similarity of the triangles OAC and aOd it follows that

$$\frac{CA}{OC} = \frac{Od}{ad} \quad \text{or} \quad \frac{CA}{OC} = \frac{T_1}{P_1}.$$

From the similarity of the triangles OCB and Ocb it follows that

$$\frac{CB}{OC} = \frac{cb}{Oc} \quad \text{or} \quad \frac{CB}{OC} = \frac{T_2}{P_2}.$$

Dividing the first proportion by the second and taking into account that $T_1 = T_2$, we obtain

$$\frac{CA}{CB} = \frac{P_2}{P_1}. \tag{18}$$

Formula (18) may also be given as follows:

$$\frac{P_1}{CB} = \frac{P_2}{CA} = \frac{P_1 - P_2}{CB - CA} = \frac{R_P}{AB}. \tag{19}$$

We thus obtain the following rule: *the resultant of two parallel forces not equal in magnitude and acting in opposite directions is parallel to the forces, acts in the direction of the greater force, and is equal to the magnitude of their difference; the line of action of the resultant lies beyond the greater force at distances from the lines of action of the component forces that are inversely proportional to the magnitudes of the forces.*

Problem 14. Determine the magnitude and the position of the resultant of two parallel forces $P_1 = 10$ N and $P_2 = 25$ N acting in opposite directions (Fig. 43) if the distance between their lines of action is $l = 30$ cm.

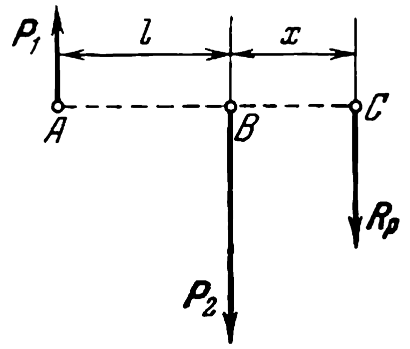


FIG. 43

Solution. The magnitude of the resultant $R_P = P_2 - P_1 = 25 - 10 = 15$ N. The resultant force lies beyond the greater force and acts in the direction of the greater force. We denote the distance of the line of action of the resultant $\mathbf{R}_P$ from the line of action of the greater force by x.

By the rule for the composition of two parallel forces acting in opposite directions [formula (19)], we have

$$\frac{P_1}{x} = \frac{R_P}{l}, \quad \text{whence} \quad x = \frac{P_1 l}{R_P} = \frac{10 \times 30}{15} = 20 \text{ cm}.$$

Sec. 20. Resolving a force into two parallel component forces

Resolution (or decomposition) is the inverse operation of composition (adding), and it may be carried out via the formulas established in the preceding sections. When resolving a force into two component forces parallel to it, both in the case where the components act in the same direction and in the case where they act in opposite directions, we have two equations [formulas (15), (16) or (17), (18)] involving four unknowns: the magnitudes of the two components and the distances of their lines of action from the line of action of the resultant. Therefore, in any general statement, this problem, like the problem of resolving a force into converging components, is indeterminate. To make the problem determinate, we need two more conditions.

For example, we have to know: (1) the distances of the lines of action of the desired components $\mathbf{P}_1$ and $\mathbf{P}_2$ up to the line of action of the given force, or (2) the magnitude of one of the component forces and the distance of its line of action from the line of action of the given force, or (3) the magnitude of one of the required component forces and the distance of the line of action of the given force from the line of action of the other desired component force. Let us consider the first case, which occurs most frequently.

Suppose it is required to decompose the given force $\mathbf{P}$ applied to a point C (Fig. 44) into two component forces acting along the straight lines I and II, which are parallel to the line of action of the given force. Draw through C some straight line that intersects the first and second lines at points A and B. These points may be taken for the points of application of the desired components $\mathbf{P}_1$ and $\mathbf{P}_2$. Since $\mathbf{P}$ must be the equivalent of the parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$, the latter two must satisfy the following equation:

$$P_1+P_2=P \quad \text{and} \quad \frac{P_1}{P_2}=\frac{CB}{AC}=\frac{b}{a}.$$

Solving this system of equations, we find the magnitudes P_1 and P_2 of the desired component forces, the lines of action of which, a and b, are given.

The solution is exactly the same in the case where the point of application of the given force **P** does not lie between the lines of action of the given components, but beyond one of them. In this case the desired components will act in opposite directions and the force **P** will be equal in magnitude to the magnitude of their difference.

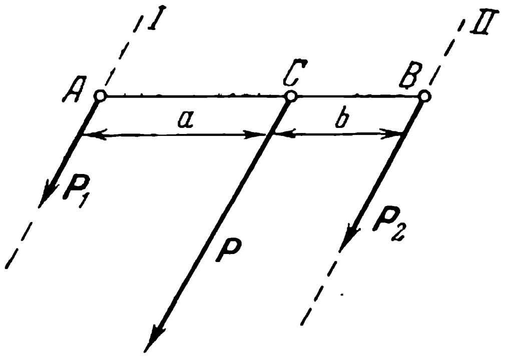


FIG. 44

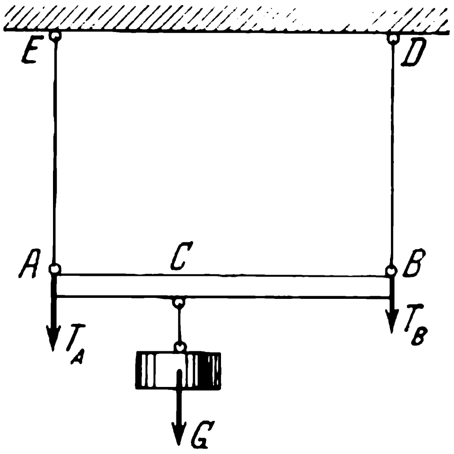


FIG. 45

Problem 15. Suspended at C from a rod AB (Fig. 45), which hangs on two parallel ropes AE and BD, is a load of weight $G = 80$ N. Determine the tension of the ropes if $AC = 30$ cm and $BC = 50$ cm. Disregard the weight of the rod.

Solution. Since the ropes are parallel to the force of gravity **G**, the problem reduces to resolving that force into two parallel components applied to the points A and B. Denoting the tension of the ropes by $\mathbf{T}_A$ and $\mathbf{T}_B$ respectively, we find, via formula (16a),

$$\frac{T_A}{BC} = \frac{T_B}{AC} = \frac{G}{AB},$$

whence we obtain

$$T_A = \frac{G \times BC}{AB} = \frac{80 \times 50}{80} = 50 \text{ N},$$

$$T_B = \frac{G \times AC}{AB} = \frac{80 \times 30}{80} = 30 \text{ N}.$$

Problem 16. A beam (Fig. 46) rests on two posts A and B. It is required to suspend a load $G = 30$ kN. The distance between the posts is $l = 6$ m. At what point C is the weight to be suspended so that the load on the weaker post A does not exceed 6 kN?

Solution. If we take the pressure on A to be $P_1 = 6$ kN, then the pressure P_2 on the post B is found from $G = P_1 + P_2$, whence $P_2 = G - P_1 = 30 - 6 = 24$ kN.

We thus know the magnitudes of the parallel components into which the force G is decomposed. In that case, the distance x of the point of application of the resultant G from the point A is found from the proportion

$$\frac{P_2}{x}=\frac{G}{l},$$

whence

$$x=\frac{lP_2}{G}=\frac{6\times 24}{30}=4.8 \text{ m}.$$

Problem 17. Resolve the force $P = 300$ N into two parallel component forces $\mathbf{P}_1$ and $\mathbf{P}_2$, one of which, $P_1 = 420$ N, acts in a direction opposite to **P** and its line of action passes at a distance $l = 6$ m from the line of action of the given force (Fig. 47).

Solution. Since the forces **P** and $\mathbf{P}_1$ act in different directions, we clearly have to do with decomposition of a force into two parallel components acting in opposite directions. The resultant of two such forces passes outside those forces beyond the greater force and in the direction of the greater force. Hence, the desired second component $\mathbf{P}_2$ must act in the same direction as the force **P**. Denote by x the distance between the lines of action of that force and the given force **P**. This distance, and also the magnitude of the force $\mathbf{P}_2$, is found from equations (17) and (18):

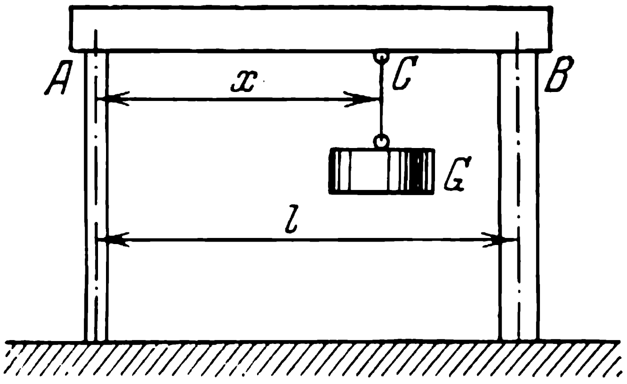


FIG. 46

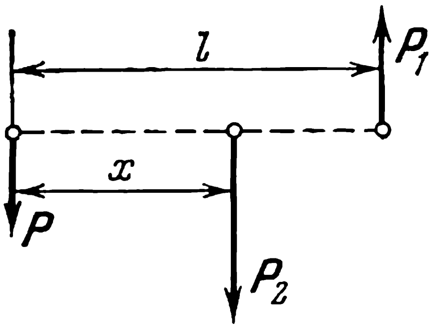


FIG. 47

$$P=P_2-P_1 \quad \text{and} \quad \frac{x}{l}=\frac{P_1}{P_2}.$$

Solving these equations, we obtain

$$P_2 = P + P_1 = 300 + 420 = 720 \text{ N},$$

$$x=\frac{lP_1}{P_2}=\frac{6\times 420}{720}=3.5 \text{ m},$$

CHAPTER IV

THE THEORY OF COUPLES IN A PLANE. THE MOMENT OF A FORCE RELATIVE TO A POINT

Sec. 21. Couples

In order to be able, on the basis of the axioms of statics, to derive the rules for the composition of two parallel forces acting in the same direction and also in opposite directions, we replace the parallel forces by equivalent systems of converging forces.

If two oppositely directed parallel forces are equal in magnitude, no such replacement is possible.

Quite true, as is evident from a glance at Fig. 42; if the magnitudes of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal, then by adding any two forces $\mathbf{T}_1$ and $\mathbf{T}_2$ equal in magnitude and opposite in direction, we will always get parallel but not converging forces $\mathbf{R}_1$ and $\mathbf{R}_2$.

The fact that the rule established in Sec. 19 for finding the resultant of two parallel forces acting in opposite directions is inapplicable in the case where these forces are equal in magnitude may also be established with the aid of an equation that follows from formula (19):

$$CA = \frac{P_2 \times AB}{R_P}.$$

If the magnitude of the force $\mathbf{P}_2$ unboundedly approaches the magnitude of the force $\mathbf{P}_1$, then the magnitude of the resultant of these forces $R_P = P_1 - P_2$ will tend to zero and the point C of its application will go to infinity because the distance AC increases without bound in the process.

From this it follows that actually there is no resultant force of such a system of forces, nor is there any point at a finite distance where that resultant could be applied.

The geometrical sum of two equal and oppositely directed forces is always equal to zero, but two such forces balance (on the basis of the first axiom of statics) only if they act along a single straight line, but in this given case they have different lines of action.* Since in all other cases two parallel forces (and forces converging to one point as well) can always be replaced by a single resultant, the given system of forces occupies a very special place among all systems and has a special name. *A system of two parallel forces equal in magnitude and opposite in direction is termed a couple.*

A couple has no resultant and cannot be balanced by any one force. This follows from the fact that if a couple were balanced by a single force, then it would have a resultant force. On the basis of the second corollary to the axioms of statics, a balancing force taken in the reverse direction would be the resultant of the given couple.

Experience shows that a couple acting on a rigid body tends to impart to that body a rotational motion if the constraints imposed on the body do not prevent this. A couple may be found in every case where the applied forces tend to impart to the body a rotational motion.

For instance, in order to impart rotational motion to the screw of the press shown in Fig. 48, forces **P** and **P′** are applied to the handle; these forces must be equal in magnitude, parallel and in opposite directions. Also, as is evident from everyday experience, for forces applied as shown in Fig. 48 at the points *A* and *B* a smaller magnitude is required to turn the press than if they were applied to points *C* and *D* closer to each other.

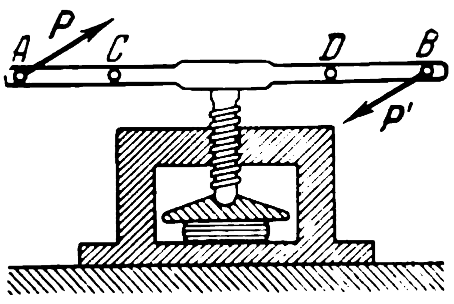


FIG. 48

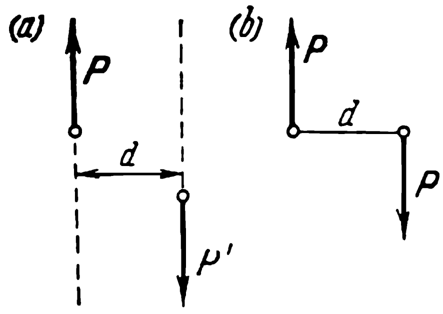


FIG. 49

---

* It is important to bear in mind for the sequel that in general the fact of the geometrical sum of forces being zero is not sufficient for the equilibrium of those forces (it is fully sufficient only for a system of converging forces).

Thus the rotatory effect of a couple depends both on the magnitude of the forces and also on the distance between the lines of their action and is defined as the so-called moment of the couple.

The magnitude of the moment of a couple is equal to the product of the magnitude of one of the forces of the couple by the arm of the couple, that is, the shortest distance between the lines of action of the forces of the couple (Fig. 49*a*).

Since a force may be translated to any point along its line of action, we will henceforth always denote the forces of a couple as in Fig. 49*b*, with the straight line joining their points of application being perpendicular to the lines of action of the forces, that is, so that at the same time it is the arm of the couple.

The plane in which the given couple is located is termed the plane of action of that couple.

The action of a couple on a body depends on: (1) the absolute numerical value of the moment of the couple, (2) the position in space of the plane of action of the couple, and (3) the direction of rotation of the couple in the plane of its action. For example, if the forces acting on the handle of the press shown in Fig. 48 were in opposite directions, then instead of going in and compressing the body, the screw (given a right-handed thread) would be going out.

For a plane system of forces, that is, when the lines of action of all forces applied to the body lie in the same plane, there is no need to indicate the position of the plane of action of the couples in the system, and the *moment of a couple is regarded as a scalar algebraic quantity.*

The algebraic magnitude of the moment of a couple is the absolute value of the moment of the couple taken with the plus or minus sign. Denoting the algebraic magnitude of the moment of a couple by the letter m and the arm of the couple by the letter d, we have

$$m = \pm P \times d. \tag{20}$$

Here the moment of the couple is taken to be positive if the couple tends to rotate a body in the counterclockwise direction, *and to be negative if the couple tends to turn the body in the clockwise direction.* For instance, the couple (**P**, **P**′)[*] shown in Fig. 49 has a negative moment, whereas the couple ($\mathbf{P}_1$, $\mathbf{P}_1'$) shown in Fig. 50 has a positive moment.

The moment of a couple obviously has the dimensions of force multiplied by the dimensions of length. Thus, if the force is measured in newtons and the arm in metres, then the moment of the couple will be measured in newton-metres (N·m).

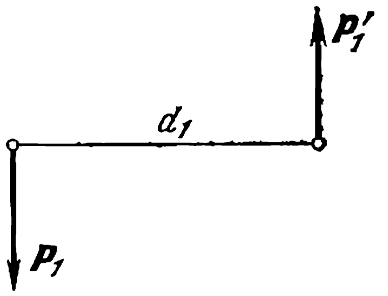


FIG. 50

Sec. 22. The moment of a force about a point (centre)

The notion of the moment of a force about a point is connected with the problem of a lever.

Referring to Fig. 51, let a force **P** be applied to the bar, the force lying in a plane perpendicular to the fixed axis of rotation of the bar. From experience we know that the bar will rotate.

We mentioned earlier (Sec. 21) that in all cases where the body is in rotational motion, there exists a couple that generates that rotation.

The bar will exert pressure at some point *O* on the fixed axis with a force **P**′ equal in magnitude and parallel to the force **P**. Now, action is always equal to reaction and therefore there will be a force **P**″ exerted on the bar by the axis at that point as well. This force is equal in magnitude to the force **P**′. Thus, the bar will be acted upon by the couple (**P**, **P**″) which will set up a rotation about the fixed axis. The moment of that couple is called the *moment of the force* **P** *about the point O.*

Since the *moment of a force about a point is actually the moment of a couple* that arises in the rotation of a body about that point, it is determined by rules that are exactly like the rules for the moment of a couple.

---

[*] Couples are commonly denoted by enclosing the forces in parentheses.

The *algebraic magnitude of the moment of a force about some point is equal to the product, taken with a plus or minus sign, of the magnitude of the force by its arm*, that is, by the length of the perpendicular dropped from that point on the line of action of the force.

The concept of the moment of a force about a point is one of the most important concepts of mechanics. By generalizing

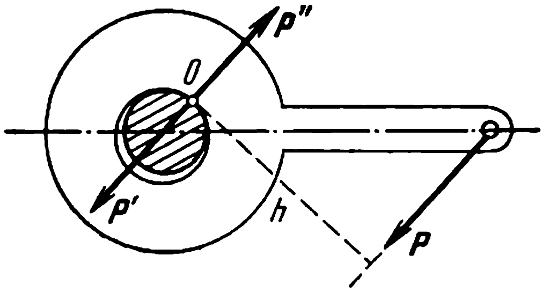


FIG. 51

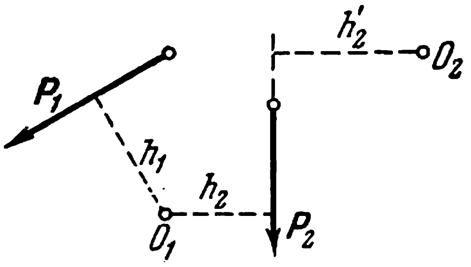


FIG. 52

hat concept it is possible to find the moment of a force bout any point, irrespective of whether the body can actualy rotate about that point or not.

The point about which one determines the moment of a force is often called the centre of the moment.

Denoting the moment of a force **P** about a centre O by the symbol $m_O(\mathbf{P})$ and the arm about the given centre by h, we have

$$m_O(\mathbf{P}) = \pm Ph. \tag{21}$$

The sign in this formula is chosen on the basis of the following rule, which is similar to the rule governing signs for the algebraic magnitude of the moment of a couple: *if the force tends to turn the body about the centre of moments in counterclockwise sense, the moment is positive, if it is turned clockwise, the moment is negative.*

Thus, in Fig. 51 the moment of the force **P** about O is

$$m_O(\mathbf{P}) = m(\mathbf{P}, \mathbf{P}'') = -Ph.$$

For the force $\mathbf{P}_1$ (Fig. 52) we have $m_{O_1}(\mathbf{P}_1) = P_1h_1$, for the force $\mathbf{P}_2$ we have $m_{O_1}(\mathbf{P}_2) = -P_2h_2$. Note that the moment of one and the same force can be both positive and negative depending on the mutual positions of the force and the centre of moments, that is, the point about which the moment is taken. For example, in Fig. 52 the moment of the force $\mathbf{P}_2$

about O_1 is equal to $m_{O_1}(\mathbf{P}_2) = -P_2h_2$ and the moment of the same force about O_2 is equal to $m_{O_2}(\mathbf{P}_2) = P_2h_2'$.

Also note that the moment of a force about a point like the moment of a couple, may be taken for the scalar algebraic quantity only in problems that deal with forces lying in a single plane.*

The moment of a force clearly has the dimensions of force multiplied by length, that is, it is measured in the same units as the moment of a couple.

From the definition of the moment of a force about a point it follows that:

(1) the moment of a force about a given point remains unchanged when the force is translated along its line of action because in that case neither the magnitude of the force nor its arm undergoes change**; Ж

(2) the moment of a force about a given point is equal to zero if the line of action of the force passes through that point because in that case the arm of the force is zero.

Sec. 23. Properties of couples

Theorem 1. *The algebraic magnitude of the moment of a couple is equal to the sum of the algebraic magnitudes of the moments of the forces making up the couple about any point lying in the plane of action of the given couple.*

Proof. Referring to Fig. 53, consider the couple ($\mathbf{P}$, $\mathbf{P}'$) with arm $ab = d$. The moment of the couple is $m = Pd$. For the centre of moments, take an arbitrary point O lying in the plane of action of the given couple. We then have

$$m_O(\mathbf{P}) = P \times Ob,$$

$$m_O(\mathbf{P}') = -P' \times Oa.$$

---

* In the case of forces lying in distinct nonparallel planes, the rule of signs becomes meaningless and the moments of a force about a point, like the moments of a couple, are regarded as vectors.

** It must be made clear that the arm of a force is the length of the perpendicular dropped from the centre of moments on the line of action of the force and not the length of the line segment joining the centre of moments and the point of application of the force (as is sometimes thought by students).

Combining these equalities and taking into account that $\mathbf{P} = \mathbf{P}'$ and $ab = d$, we obtain

$$m_O(\mathbf{P}) + m_O(\mathbf{P}') = P \times Ob - P' \times Oa = P(Ob - Oa)$$
$$= P \times ab = Pd = m(\mathbf{P}, \mathbf{P}').$$

Consequently, the sum of the moments of the forces of a couple does not depend on the choice of centre of the moments. It is equal to a quantity that is constant for the given couple: it is equal to the moment of the couple and that moment describes the rotational action of the couple on the body.

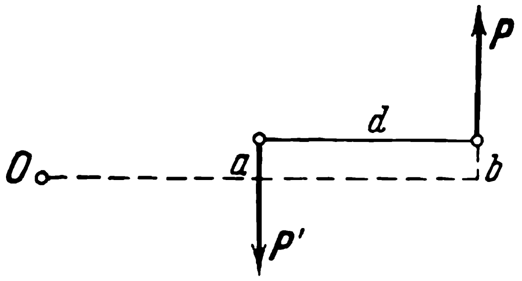


FIG. 53

Theorem 2. *Any couple may, without altering its action on an absolutely rigid body, be replaced by another couple located arbitrarily in the same plane and having the same algebraic magnitude of the moment as the given couple.*

Proof. Let a body be acted on by a couple ($\mathbf{P}_1$, $\mathbf{P}_1'$) with arm $AB = d$ (Fig. 54). We apply to points A and B two forces

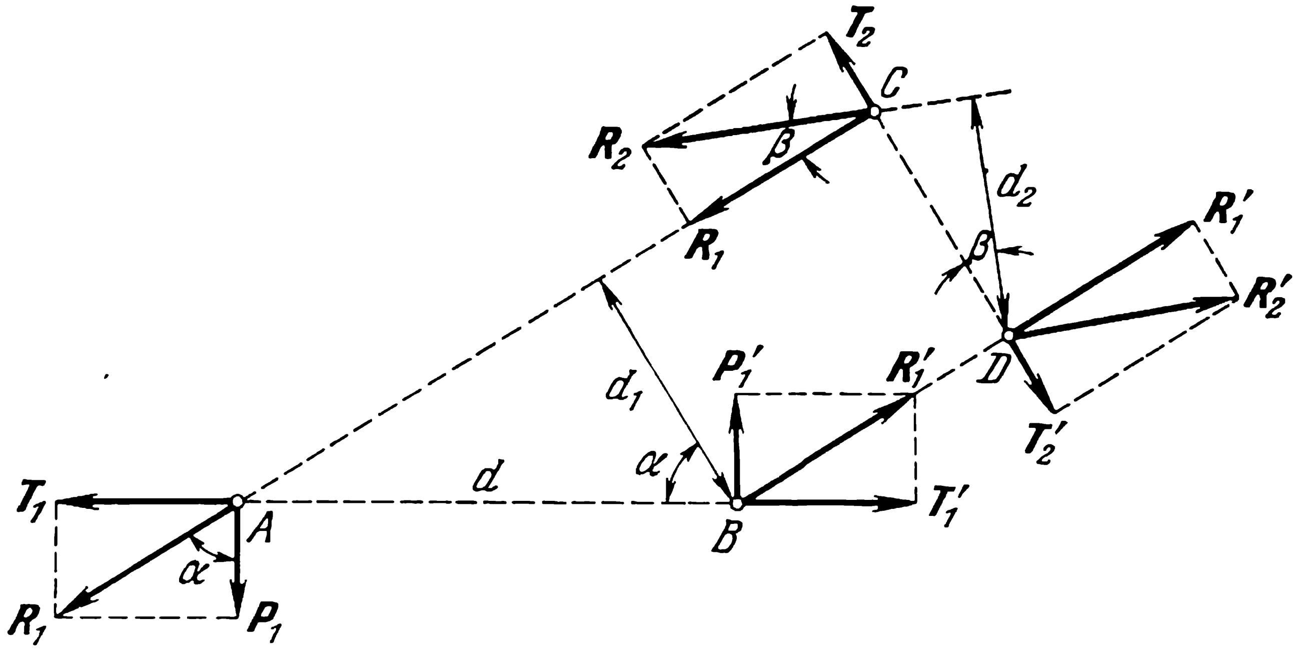


FIG. 54

$\mathbf{T}_1$ and $\mathbf{T}_1'$ that are equal in magnitude and act in opposite directions along the same straight line. Adding the forces $\mathbf{P}_1$ and $\mathbf{T}_1$ and the forces $\mathbf{P}_1'$ and $\mathbf{T}_1'$ pairwise, we clearly arrive at a new couple, ($\mathbf{R}_1$, $\mathbf{R}_1'$). Since the system of forces ($\mathbf{T}_1$, $\mathbf{T}_1'$)

is balanced, the resulting couple ($\mathbf{R}_1$, $\mathbf{R}_1'$) is equivalent to the given couple.

If the forces of that couple are translated to any other two points lying on the lines of their action (say, *C* and *D* in Fig. 54), and if we again join to the forces of the given couple any balanced system of forces $\mathbf{T}_2$, $\mathbf{T}_2'$, we get the couple ($\mathbf{R}_2$, $\mathbf{R}_2'$), which is equivalent to the given couple:

$$(\mathbf{P}_1, \mathbf{P}_1') \sim (\mathbf{R}_1, \mathbf{R}_1') \sim (\mathbf{R}_2, \mathbf{R}_2').$$

It is easy to see that all the resulting equivalent couples have the same sense of rotation (counterclockwise here) and the same absolute value of the moment.

Indeed, as is evident from Fig. 54, $R_1 = P_1/\cos\alpha$, $d_1 = d\cos\alpha$ and, hence, $R_1d_1 = P_1d$,

$$R_2 = \frac{R_1}{\cos\beta}, \quad d_2 = d_1\cos\beta \quad \text{and} \quad R_2d_2 = R_1d_1.$$

By translating the forces of the couple along their lines of action and by repeating operations similar to those given above, it is clearly possible to translate the couple to any position in the plane of its action and to alter in any way the magnitudes of the forces of the couple. Then, accordingly, there will be a change in the length of the arm of the couple, but the absolute value of the moment of the couple and its sense of rotation will remain unchanged.

Translating a couple in its plane of action is often made use of in practical problems. For example, the driver of an automobile turns the steering wheel with both hands and the same force, thus acting on it with a couple. In doing so, he can grasp the steering wheel at any two diametrically opposite points: the action of the couple remains unchanged, and the same effort is required in all cases.

One must bear in mind that the translation of a couple in its plane of action, just like the translation of a force along its line of action, is applicable without condition only to absolutely rigid bodies. We can use this property of a couple when solving problems involving the equilibrium of external forces applied to a deformed body as well, since this equilibrium is not upset if the body is thought to be absolutely rigid (the principle of rigidity). However, the deformation of a body and the internal forces that result from such deformation and that counteract that deformation depend on the location of the couple; for this reason, strength of materials problems always indicate the cross-section of the body on which a couple operates.

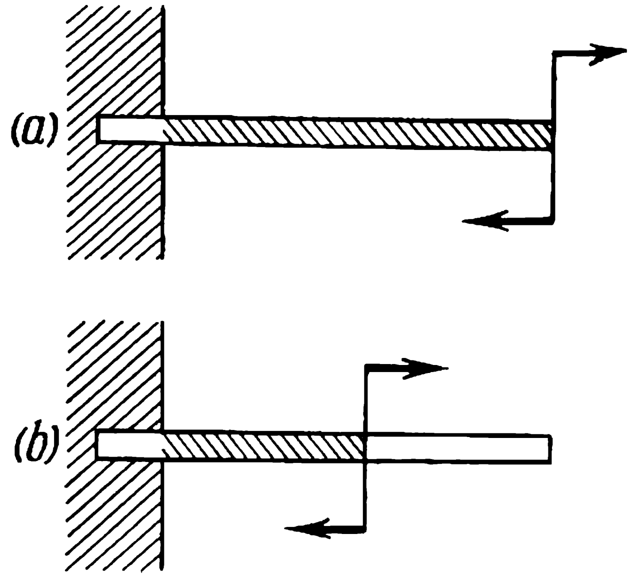


FIG. 55

By way of an illustration, Fig. 55 depicts two beams, one end of each being fixed in the wall and with couples acting. It is clear that the couple applied to the end cross-section (Fig. 55*a*) will deform (bend) the whole beam, whereas the couple applied to the middle cross-section (Fig. 55*b*) will bend only the left-hand portion of the beam.

From the theorem just proved it follows that the *action of a couple on a body is fully determined by its moment.*

Infinitely many distinct couples correspond to the same moment, but all couples with the same moment are equivalent as to their mechanical effect and may replace one another. For this reason, problems in mechanics ordinarily specify the moment of a couple and not the magnitudes of the forces of the couple and the arm. Frequently, a couple is depicted as in Fig. 56 by an arrow indicating the sense of rotation without any indication of the forces of the couple or of its arm. The arrow is labelled with the modulus (magnitude) *m* of the moment.

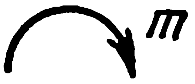


FIG. 56

Sec. 24. The composition of couples. The condition of equilibrium of couples

As we have already pointed out, a couple is a system of forces that does not lend itself to any further simplification, that is, it cannot be replaced by a single force. Therefore in statics, a couple, like a single force, is regarded as an independent, irreducible element. Like forces, couples may be combined and decomposed (resolved). A couple that replaces the action on a body of all given couples taken together is termed the *resultant couple*, and the given couples are *its components*.

Theorem. *Several couples lying in the same plane may be replaced by a single resultant couple whose moment is equal to the algebraic sum of the moments of the component couples.*

Proof. Suppose a body is acted upon by three couples arbitrarily located in the plane of the drawing with moments m_1, m_2, and m_3. The senses of rotation of the couples are indicated in Fig. 57*a* by arrows. For the arm we take an arbitrary segment $AB = d$ and reduce all given couples to that arm, that is, we replace the given couples by equivalent couples $(\mathbf{P}_1, \mathbf{P}'_1)$, $(\mathbf{P}_2, \mathbf{P}'_2)$ and $(\mathbf{P}_3, \mathbf{P}'_3)$ with arm *d*. The magnitudes of the forces of the couples are determined from the condition that the moments of equivalent couples are equal:

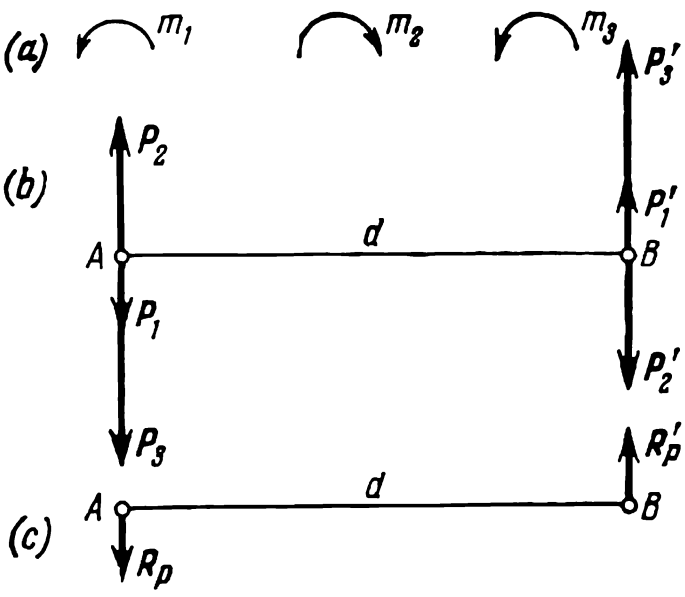


FIG. 57

$$P_1 = \frac{|m_1|}{d}, \quad P_2 = \frac{|m_2|}{d} \quad \text{and} \quad P_3 = \frac{|m_3|}{d}.$$

Using the fact that a couple may be translated to any position in its plane, translate the new couples in the plane of the drawing so that their arms coincide with segment AB. Then all forces will be located on two straight lines perpendicular to AB and passing through its end points A and B (Fig. 57*b*). Combining all the forces acting along the straight line that passes through the point A, we obtain the resultant, the magnitude of which is $R_P = P_1 + P_3 - P_2$. In the same way, combining all forces acting along the straight line passing through B, we obtain the second resultant whose magnitude is $R'_P = P'_1 + P'_3 - P'_2$.

The forces $\mathbf{R}_P$ and $\mathbf{R}'_P$ clearly constitute a couple. Thus, the three given couples are reduced to a single resultant couple $(\mathbf{R}_P, \mathbf{R}'_P)$, as shown in Fig. 57*c*.

The moment of the resultant couple is

$$m(\mathbf{R}_P, \mathbf{R}'_P) = R_P d = (P_1 + P_3 - P_2)\, d$$
$$= P_1 d + P_3 d + (-P_2 d) = m_1 + m_3 + m_2.$$

The moment m_2 of the second couple is negative since its sense of rotation is clockwise. Consequently,

$$m(\mathbf{R}_P, \mathbf{R}'_P) = \sum m(\mathbf{P}_k, \mathbf{P}'_k). \tag{22}$$

As is evident from the development of the proof, this theorem holds for any number of component couples.

Corollary. *For the equilibrium of couples located in the same plane it is necessary and sufficient that the algebraic sum of the moments of all given couples be zero; that is, the condition of equilibrium of couples is*

$$\sum m(\mathbf{P}_k, \mathbf{P}'_k) = 0. \tag{23}$$

True enough, a rigid body acted upon by a single couple is in a state of equilibrium if the forces of the couple are zero, or the arm of the couple is zero; that is, the couple is made up of two forces equal in magnitude and acting along a single straight line in opposite directions. In both cases, the moment of the couple is equal to zero. Therefore, for a body acted upon by a single couple to be in equilibrium, it is necessary and sufficient that the moment of the couple be zero. But, as was just demonstrated, any plane system of couples may be replaced by a single couple whose moment is equal to the algebraic sum of the moments of the given couples, whence follows the corollary.

Problem 18. Five couples are acting in a single plane. The direction of rotation of three couples $(\mathbf{P}_1, \mathbf{P}'_1)$, $(\mathbf{P}_2, \mathbf{P}'_2)$ and $(\mathbf{P}_3, \mathbf{P}'_3)$ with arms equal to 0.75 m, 0.4 m and 0.2 m respectively is clockwise, the sense of rotation of the other two couples $(\mathbf{P}_4, \mathbf{P}'_4)$, and $(\mathbf{P}_5, \mathbf{P}'_5)$ with arms 0.2 m and 0.5 m is in the opposite direction. $P_1 = 2$ N, $P_2 = 5$ N, $P_3 = 15$ N, $P_4 = 35$ N and $P_5 = 12$ N. Find the moment of the resultant couple and also the magnitudes of its forces if the arm is made equal to 0.5 m.

Solution. The moment of the resultant couple is equal to the algebraic sum of the moments of the component couples [formula (22)]:

$$m(\mathbf{R},\ \mathbf{R}') = \sum m(\mathbf{P}_k,\ \mathbf{P}'_k)$$

$$= -2\times 0.75-5\times 0.4-15\times 0.2+35\times 0.2+12\times 0.5=6.5\ \text{N}\cdot\text{m}.$$

The signs of the moments are taken in accord with the rule given earlier. Since the resultant couple has a positive moment, its rotation is counterclockwise. Reducing the resultant couple to the arm $d = 0.5$ m, we get the magnitude of its forces:

$$R=R'=\frac{m(\mathbf{R},\ \mathbf{R}')}{d}=\frac{6.5}{0.5}=13\ \text{N}.$$

Problem 19. The lengths of overhang of a beam (Fig. 58) are acted upon by two equal and parallel forces $P = P' = 30$ kN, the force **P** being directed upwards and the force **P**′ downwards.

Determine the reactions of the supports of the beam, disregarding its weight, if the beam span $l = 6$ m and the length of each overhang is $a = 2$ m.

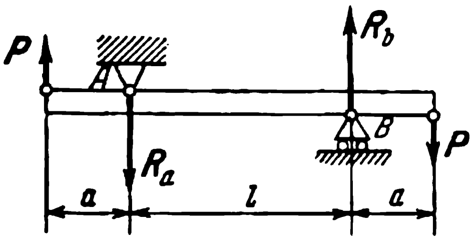


FIG. 58

Solution. The given beam with overhang is acted upon by the couple (**P**, **P**′) that tends to turn the beam clockwise. The moment of this couple is $m(\mathbf{P}, \mathbf{P}') = P(l + 2a)$. Since the beam is in a state of equilibrium and a couple can be balanced only by a couple, the reactions of the supports, $\mathbf{R}_a$ and $\mathbf{R}_b$, must also constitute a couple ($\mathbf{R}_a$, $\mathbf{R}_b$) that tends to turn the beam in the opposite direction, that is, counterclockwise. The moment of this couple of reaction forces $m(\mathbf{R}_a, \mathbf{R}_b) = R_a l$. According to the condition of the equilibrium of couples (23),

$$m(\mathbf{P},\ \mathbf{P}')+m(\mathbf{R}_a,\ \mathbf{R}_b)=-P(l+2a)+R_a l=0$$

whence

$$R_b=R_a=\frac{P(l+2a)}{l}=\frac{30(6+2\times 2)}{6}=50\ \text{kN}.$$

CHAPTER V

FORCES ARBITRARILY LOCATED IN A PLANE

Sec. 25. Poinsot's* theorem on the parallel translation of a force

A force may be translated parallel to its original direction to any point of a body without altering the action of the force on the body by adjoining some couple.

Proof. Suppose we have a force **P** applied to a body at a point A (Fig. 59). Take an arbitrary point O of that body and apply to it two oppositely directed forces **P**′ and **P**″ that are parallel to the given force **P** and equal to it in magnitude. The forces **P**′ and **P**″ mutually balance and therefore the action on the body of the single given force **P** is equivalent to the action on it of the system of three forces **P**′, **P**″, and **P**. The force **P**′ may be regarded here as the force **P** translated parallel to its original direction to the point O, while the forces **P**″ and **P** form a couple that we must adjoin in the parallel translation of the force from A to O (so that the action of the force on the body remains unchanged in the translation). The proof is complete.

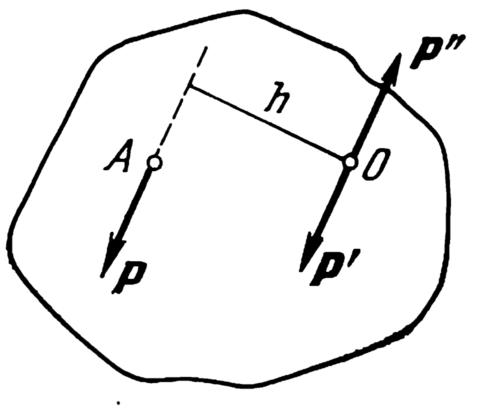


FIG. 59

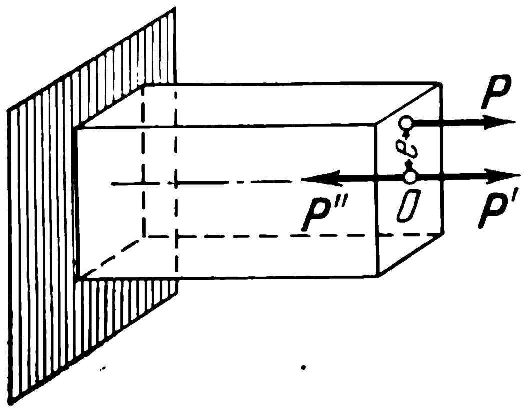


FIG. 60

The converse is quite possible: *a force and a couple lying in the same plane may always be replaced by a single force*

---

* Poinsot, Louis (1777-1859), French mathematician.

equal to the given force translated to some point parallel to its original direction.

Indeed, a given force can always be translated parallel to its original direction to some point so that an adjoined couple has a moment equal in absolute value but opposite in sign to the moment of the given couple. These two couples then mutually balance and what is left is a single force equal in magnitude to the given force and in the same direction but having a different line of action.

The parallel translation of a force is not only a highly fruitful formal procedure but, in a number of cases, corresponds to the physical essence of the phenomena as well. By way of an illustration, let us consider a bar (Fig. 60) acted upon by a force **P** applied perpendicularly to a cross-section of the bar at a distance e from the centre of gravity O of that cross-section.

Apply to point O a balanced system of forces $(\mathbf{P}', \mathbf{P}'')$ such that $P' = P'' = P$. Then the action on the bar of the given (off-centre) force **P** is equivalent to the action on it of the force $\mathbf{P}'$ (that passes through the centre O and stretches the bar) and of the couple $(\mathbf{P}, \mathbf{P}'')$ with moment $m(\mathbf{P}, \mathbf{P}'') = m_O(\mathbf{P}) = -P \times e$ that bends the bar.

Sec. 26. Reducing a plane system of forces to a single centre. The principal vector and the principal moment

Suppose we have a system of several* (say, four) forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ located in arbitrary fashion in a plane (Fig. 61). In the plane of action of the forces, take an arbitrary point O. Call that point the centre of reduction and in turn reduce to it (using the Poinsot theorem) all the given forces. As the result of the reduction, we obtain a system of forces $\mathbf{P}_1'$, $\mathbf{P}_2'$, $\mathbf{P}_3'$, and $\mathbf{P}_4'$ applied to that point and a system of couples:

$$(\mathbf{P}_1, \mathbf{P}_1''), \ (\mathbf{P}_2, \mathbf{P}_2''), \ (\mathbf{P}_3, \mathbf{P}_3'') \text{ and } (\mathbf{P}_4, \mathbf{P}_4'').$$

---

* There can be any number of forces making up the system. The subsequent reasoning and derivation do not depend on the number of forces involved.

The forces $\mathbf{P}_1'$, $\mathbf{P}_2'$, $\mathbf{P}_3'$ and $\mathbf{P}$ applied to the point O may be combined via the force-polygon rule and, consequently, may be replaced by a single equivalent force $\mathbf{R}_{\text{pr}}$ equal to

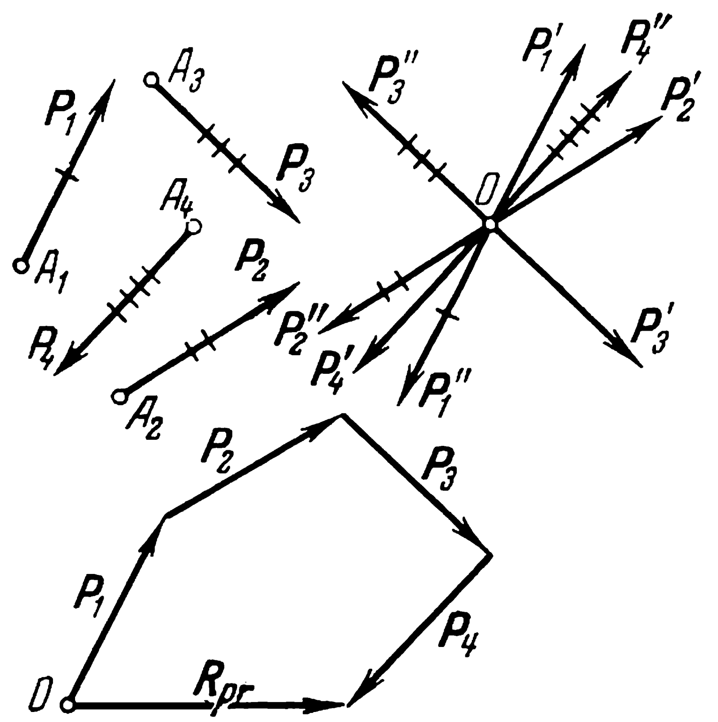


FIG. 61

their geometric sum. Since the forces $\mathbf{P}_1'$, $\mathbf{P}_2'$, $\mathbf{P}_3'$, and $\mathbf{P}_4'$ are geometrically equal to the given forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$, it follows that

$$\mathbf{R}_{\text{pr}} = \mathbf{P}_1' + \mathbf{P}_2' + \mathbf{P}_3' + \mathbf{P}_4' = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 = \sum \mathbf{P}_k.$$

The vector $\mathbf{R}_{\text{pr}}$, *which is equal to the geometric sum of all forces of the given system, is the principal vector of that system*:

$$\mathbf{R}_{\text{pr}} = \sum \mathbf{P}_k. \tag{24}$$

The magnitude and direction of the principal vector may be found using formulas (11) and (12) of an equivalent system of converging forces:

$$R_{\text{pr}} = \sqrt{R_{\text{pr}\,x}^2 + R_{\text{pr}\,y}^2} = \sqrt{\left(\sum X_k\right)^2 + \left(\sum Y_k\right)^2},$$

$$\cos(\widehat{\mathbf{R}_{\text{pr}}, x}) = \frac{R_{\text{pr}\,x}}{R_{\text{pr}}}, \quad \cos(\widehat{\mathbf{R}_{\text{pr}}, y}) = \frac{R_{\text{pr}\,y}}{R_{\text{pr}}}.$$

All adjoined couples—$(\mathbf{P}_1, \mathbf{P}_1'')$, $(\mathbf{P}_2, \mathbf{P}_2'')$, $(\mathbf{P}_3, \mathbf{P}_3'')$, $(\mathbf{P}_4, \mathbf{P}_4'')$ may be combined by the rule of combining couples lying in one plane and, hence, may be replaced by a single resultant couple. The moments of these couples are equal (according to what was said in Sec. 22) to the moments of the given forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ about the centre of reduction O, that is,

$$\begin{aligned} m(\mathbf{P}_1, \mathbf{P}_1'') &= m_O(\mathbf{P}_1), \\ m(\mathbf{P}_2, \mathbf{P}_2'') &= m_O(\mathbf{P}_2), \\ m(\mathbf{P}_3, \mathbf{P}_3'') &= m_O(\mathbf{P}_3), \\ m(\mathbf{P}_4, \mathbf{P}_4'') &= m_O(\mathbf{P}_4). \end{aligned}$$

From this, denoting the moment of the resultant couple by M_O, we obtain

$$\begin{aligned} M_O &= m(\mathbf{P}_1, \mathbf{P}_1'') + m(\mathbf{P}_2, \mathbf{P}_2'') + m(\mathbf{P}_3, \mathbf{P}_3'') + m(\mathbf{P}_4, \mathbf{P}_4'') \\ &= m_O(\mathbf{P}_1) + m_O(\mathbf{P}_2) + m_O(\mathbf{P}_3) + m_O(\mathbf{P}_4) = \sum m_O(\mathbf{P}_k). \end{aligned}$$

The algebraic sum of the moments of all given forces, located in arbitrary fashion in the plane, about some point O is termed the principal moment of the given plane system of forces about that point:

$$M_O = \sum m_O(\mathbf{P}_k). \tag{25}$$

The result obtained from reducing to a single point a system of forces arbitrarily located in the plane may now be formulated as follows:

Any plane system of forces may always be replaced by a single force equal to the principal vector of the system and applied at an arbitrary point O, and by a couple whose moment is equal to the principal moment of the given system of forces about that point O.

The *magnitude and direction of the principal vector do not depend on the choice of the centre of reduction O*, because all forces are translated to O parallel to their original directions, and the force polygon will hence be the same in all cases. Conversely, the *numerical value and the sign of the principal moment depend*, generally, *on the choice of the centre of reduction*, because any change in the centre of reduction results in a change in the moments of the given forces about that centre and, consequently, in a change in their algebraic sum.

For this reason, when we speak of the principal moment of a given system of forces, we always indicate the point to which the moment refers.

Sec. 27. Reducing a plane system of forces to a single couple

The principal vector $\mathbf{R}_{pr}$ of a given plane system of forces will be equal to zero if the force polygon constructed for it is closed. This would be a quite sufficient condition for the equilibrium of converging forces. However, in the case of an arbitrarily located system of forces in the plane, the system is equivalent not to one single force equal to the geometric sum of the forces but to a combination of that force applied at an arbitrary centre of reduction O and a couple whose moment is equal to the principal moment M_O about the chosen centre of reduction O. Therefore if the principal vector $\mathbf{R}_{pr}$ of the given system is equal to zero, and its principal moment M_O is nonzero, then the system obviously reduces to a couple. The moment of the couple is equal to the principal moment of the given forces about the centre of reduction.

Since, by theorem 1 of Sec. 23, the moment of a couple is equal to the algebraic sum of the moments of the component forces about any point lying in the plane of action of the couple, the value of the principal moment M_O in the given case cannot depend on the choice of the centre of reduction, otherwise one and the same system of forces would be replaced by couples with different moments, which is impossible.

Hence, *if the principal vector of a given plane system of forces is zero and its principal moment is not zero, then the system is equivalent to a couple whose moment is equal to the algebraic sum of the moments of all given forces about any point of the plane.*

Problem 20. Equal (in magnitude) forces $P_1 = P_2 = P_3 = P_4 = 100$ N are acting along the sides of a square $ABCD$ in the directions indicated in Fig. 62. The square has side $a = 0.2$ m. Reduce this system to the point A.

Solution. Draw the coordinate axes as shown in Fig. 62. Then project the forces on these axes and determine their moments about the centre of reduction A. Tabulate the values thus found.

Force	Projections of force on axes		Moment of force about A
	x	y	
$\mathbf{P}_1$	0	$P_1=100$ N	$P_1\times 0=0$
$\mathbf{P}_2$	$P_2=100$ N	0	$-P_2\times a=-20$ N·m
$\mathbf{P}_3$	0	$-P_3=-100$ N	$-P_3\times a=-20$ N·m
$\mathbf{P}_4$	$-P_4=-100$ N	0	$P_4\times 0=0$

The projections of the principal vector are

$$R_{\text{pr}\,x}=\sum X_k=0, \qquad R_{\text{pr}\,y}=\sum Y_k=0$$

whence the magnitude of the principal vector of the given system of forces is

$$R_{\text{pr}}=\sqrt{\left(\sum X_k\right)^2+\left(\sum Y_k\right)^2}=0.$$

The principal moment of the given system of forces about the centre of reduction A is

$$M_A=\sum m_A(\mathbf{P}_k)=-40 \text{ N·m}.$$

Since $\mathbf{R}_{\text{pr}}=0$, the given system of forces reduces to a couple with moment $M_A=-40$ N·m.

The same result could have been obtained in a different and simpler manner. From Fig. 62 it is evident that the given system of forces consists of two couples ($\mathbf{P}_1$, $\mathbf{P}_3$) and ($\mathbf{P}_2$, $\mathbf{P}_4$) that tend to turn the square clockwise. But these two couples lying in one plane may be replaced by a single resultant couple ($\mathbf{P}$, $\mathbf{P}'$) whose moment is equal to the algebraic sum of the moments of the constituent couples:

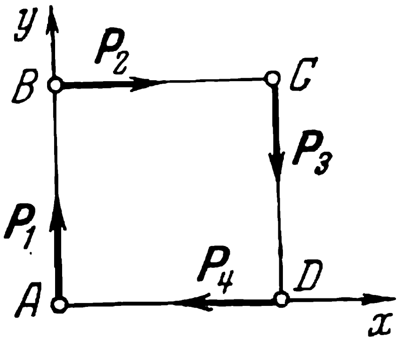


FIG. 62

$$m(\mathbf{P}, \mathbf{P}')=m(\mathbf{P}_1, \mathbf{P}_3)+m(\mathbf{P}_2, \mathbf{P}_4)$$
$$=-P_1\times a-P_3\times a=-100\times 0.2-100\times 0.2=-40 \text{ N.m}.$$

Sec. 28. Reducing a plane system of forces to a resultant

Suppose a given system of forces is reduced to some principal vector $\mathbf{R}_{\text{pr}}$ applied at an arbitrary centre of reduction O and also to some couple with a positive moment M_O (Fig. 63*a*).

Transform the couple with the given moment M_O so that the forces constituting the couple (denote them as $\mathbf{R}_P$ and $\mathbf{R}$) are equal in magnitude to the force $\mathbf{R}_{\text{pr}}$; here, one must take the arm d of the couple so that its moment remains equal to M_O. We thus obtain the couple ($\mathbf{R}_P$, $\mathbf{R}$), where $R_P = R = R_{\text{pr}}$, and the arm of the couple $d = M_O/R_{\text{pr}}$.

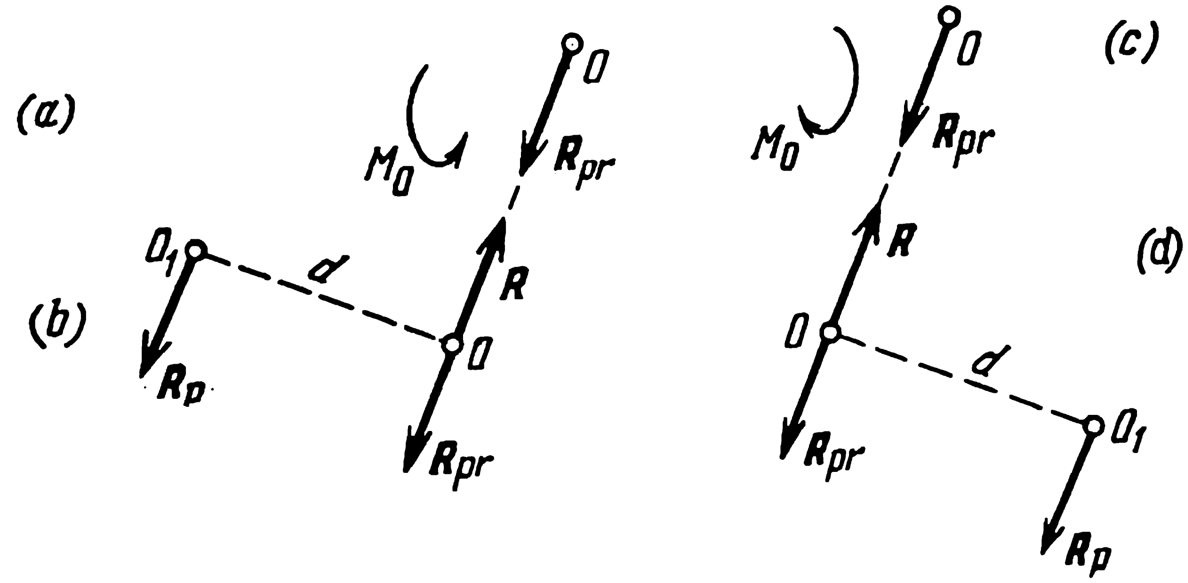


FIG. 63

By theorem 2 of Sec. 23, a couple can always be translated to any place in the plane. Translate the couple ($\mathbf{R}_P$, $\mathbf{R}$) so that the force $\mathbf{R}$ is applied to the centre of reduction O and is in a direction opposite to the principal vector $\mathbf{R}_{\text{pr}}$ (Fig. 63*b*). Then the direction of rotation of the couple must remain unchanged (for $M_O > 0$ it must be counterclockwise). The given plane system of forces is equivalent to the force $\mathbf{R}_{\text{pr}}$ and the couple ($\mathbf{R}_P$, $\mathbf{R}$), but the forces $\mathbf{R}_{\text{pr}}$ and $\mathbf{R}$ mutually balance and so there remains the single force $\mathbf{R}_P$, which, consequently, is the resultant of the given system of forces.

If the given system of forces, upon reduction to some centre O were reduced to the principal vector $\mathbf{R}_{\text{pr}}$ and to a couple with negative moment M_O, then by similar reasoning we would arrive at the resultant (Figs. 63*c* and 63*d*).

Because the couple ($\mathbf{R}_P$, $\mathbf{R}$) has in this case the opposite sense of rotation (clockwise), the resultant $\mathbf{R}_P$ is on the other side of the centre of reduction O.

It is readily seen that in both cases the *line of action of the resultant is at a distance* $d = |M_O|/R_{\text{pr}}$ *from the centre of reduction* O, *this distance being laid off in a direction so that the sign of the moment of the resultant about the centre of reduction* O *coincides with the sign of the principal moment* M_O (Figs. 63*b* and 63*d*).

If in reducing to some centre the principal moment M_O of the given system of forces about that centre is zero, but the principal vector of the system, $\mathbf{R}_{\text{pr}}$, is not zero, then the line of action of the resultant of the system will pass through the centre of reduction.

Thus, *if the principal vector of the given plane system of forces is not zero, then that system is reduced to a resultant that is equal in magnitude and direction to the principal vector* $\mathbf{R}_{\text{pr}}$.

The force $\mathbf{R}_{\text{pr}}$, which is equal to the principal vector of the system and is applied at the centre of reduction O, is not their resultant in the general case of an arbitrary location of the forces in the plane. Such a system is equivalent, generally, to the combination of the force $\mathbf{R}_{\text{pr}}$ and a certain couple. Given an arbitrary location of the forces in the plane, the system may not have a resultant and may be reducible to a couple. But if a plane system of forces does have a resultant, then that resultant is, in all cases, equal in magnitude and direction to the principal vector $\mathbf{R}_{\text{pr}}$. Here, for converging forces the line of action of the resultant passes through the common point of intersection of the forces; for forces which are located at random in the plane, the position of the line of action of the resultant is determined by the magnitude and sign of the principal moment.

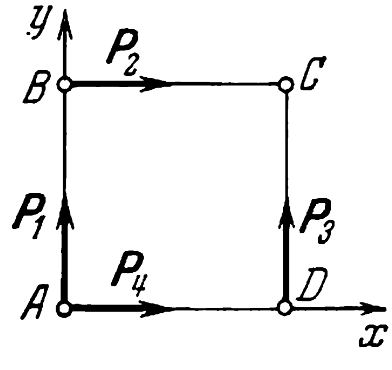


FIG. 64

Problem 21. Acting along the sides of a square $ABCD$ (Fig. 64) are forces, equal in magnitude, $P_1 = P_2 = P_3 = P_4 = 100$ N. A side of the square $a = 0.2$ m. Bring this system to the point A.

Solution. Draw the coordinate axes as shown in Fig. 64. Project the given forces on these axes and determine their moments about the centre of reduction A. Tabulate the values thus found.

Force	Projections of force on axes		Moment of force about A
	x	y	
$\mathbf{P}_1$	0	$P_1=100$ N	0
$\mathbf{P}_2$	$P_2=100$ N	0	$-P_2a=-20$ N·m
$\mathbf{P}_3$	0	$P_3=100$ N	$P_3a=20$ N·m
$\mathbf{P}_4$	$P_4=100$ N	0	0

The projections of the principal vector are

$$R_{\text{pr}\,x}=\sum X_k=200\text{ N},\qquad R_{\text{pr}\,y}=\sum Y_k=200\text{ N}.$$

The magnitude of the principal vector is

$$R_{\text{pr}}=\sqrt{\left(\sum X_k\right)^2+\left(\sum Y_k\right)^2}=\sqrt{200^2+200^2}\approx 283\text{ N}.$$

The direction of the principal vector is determined from the formula

$$\cos\left(\widehat{\mathbf{R}_{\text{pr}},\,x}\right)=\frac{\sum X_k}{R_{\text{pr}}}=\frac{200}{200\sqrt{2}}=\frac{\sqrt{2}}{2}$$

whence $(\widehat{\mathbf{R}_{\text{pr}},\,x})=45°$.

The principal moment about the centre of reduction A is

$$M_A=\sum m_A(\mathbf{P}_k)=-20+20=0.$$

Since $R_{\text{pr}}\neq 0$, and $M_A=0$, it follows that the given system reduces to the resultant $\mathbf{R}_P=\mathbf{R}_{\text{pr}}$ taken along the diagonal AC of the square and equal in magnitude to 283 N.

We would obtain the same result if we translated the forces $\mathbf{P}_2$ and $\mathbf{P}_3$ to the point C of intersection of their lines of action and then combined these forces via the parallelogram rule with the forces $\mathbf{P}_1$ and $\mathbf{P}_4$ applied at the point A. We would thus obtain two forces directed along one straight line (the diagonal AC of the square) in the same direction and such as could be replaced by a single resultant equal in magnitude:

$$R_P=2\sqrt{100^2+100^2}\approx 283\text{ N}.$$

Problem 22. Acting on a bridge truss (Fig. 65) are vertical forces $P_1=20$ kN and $P_2=40$ kN at respective distances of 10 m and 40 m from the left-hand end of the truss, and a horizontal force $P_3=30$ kN on the level of the top chord of the truss, *CD*. The truss is 6 m high. Determine the resultant of the forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$.

Solution. Taking point *C* for the centre of reduction, we reduce the given system of forces to that point. To compute the principal vector and the principal moment, draw coordinate axes x and y as shown in Fig. 65; project all forces on these axes and take their moments about the chosen centre of reduction *C*.

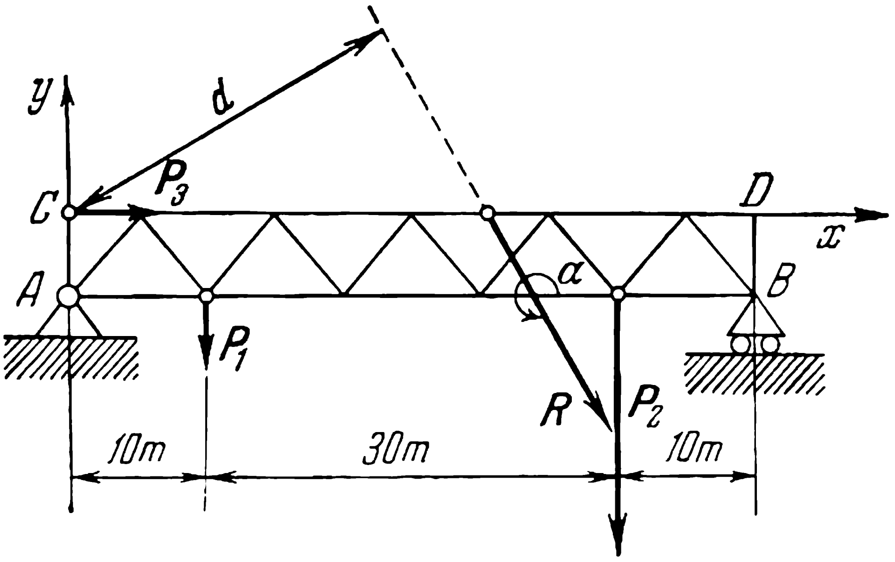


FIG. 65

Force	Projections of force on axes		Moment of force about C
	x	y	
$\mathbf{P}_1$	0	$-P_1=-20$ kN	$-P_1\times 10=-200$ kN·m
$\mathbf{P}_2$	0	$-P_2=-40$ kN	$-P_2\times 40=-1600$ kN·m
$\mathbf{P}_3$	$P_3=30$ kN	0	$P_3\times 0=0$

The projections of the principal vector $\mathbf{R}_{\text{pr}}$ on the given coordinate axes are:

$$R_{\text{pr}\,x}=\sum X_k=30\text{ kN},\qquad R_{\text{pr}\,y}=\sum Y_k=-20-40=-60\text{ kN}.$$

From this we find the magnitude of the principal vector:

$$R_{\text{pr}}=\sqrt{\left(\sum X_k\right)^2+\left(\sum Y_k\right)^2}=\sqrt{30^2+(-60)^2}=30\sqrt{5}\approx 67.1\text{ kN}.$$

Since the principal vector $\mathbf{R}_{\text{pr}}$ is nonzero, the given system of forces can be reduced to a resultant equal in magnitude and direction to the principal vector.

The direction of the resultant is determined from the formulas

$$\cos\alpha = \cos(\widehat{\mathbf{R}_P, x}) = \frac{R_{Px}}{R_P} = \frac{30}{30\sqrt{5}} = 0.447,$$

$$\sin\alpha = \cos(\widehat{\mathbf{R}_P, y}) = \frac{R_{Py}}{R_P} = -\frac{60}{30\sqrt{5}} = -0.894.$$

Since $\cos\alpha$ is positive and $\sin\alpha$ is negative, it follows that the angle $\alpha = (\widehat{\mathbf{R}_P, x})$ lies in the fourth quadrant and is equal to 296°40′.

The line of action of the resultant passes through the centre of reduction at a distance $d = |M_C|/R_{\text{pr}}$, where M_C is the principal moment of the given system of forces about the chosen centre of reduction C and is equal to $\sum m_C(\mathbf{P}_k) = -200 - 1600 = -1800$ kN·m. The sign of the moment shows that this distance d (the length of the perpendicular dropped from the centre of reduction on the line of action of the resultant) must be laid off to the right of the centre of reduction (as shown in Fig. 65).

In our case, the distance

$$d = \frac{|M_C|}{R_P} = \frac{1800}{67} = 26.9 \text{ m}.$$

Knowing the distance d and the angle $(\widehat{\mathbf{R}_P, x})$, it is easy (if need be) to find the points of intersection of the line of action of the resultant with the top chord and bottom chord of the truss.

Sec. 29. The conditions of equilibrium of an arbitrary plane system of forces. Varignon's theorem on the moment of a resultant. The condition of equilibrium of a lever

For the equilibrium of an arbitrary plane system of forces it is necessary and sufficient that both the principal vector and the principal moment of the system about any point of the plane of action of the forces be equal to zero:

$$\mathbf{R}_{\text{pr}} = 0 \quad \text{and} \quad M_O = \sum m_O(\mathbf{P}_k) = 0. \tag{26}$$

From the equality $\mathbf{R}_{\text{pr}} = 0$ it follows that the geometric sum of all forces applied at the centre of reduction is zero; therefore, by the condition of equilibrium of converging forces (see Sec. 14) these forces balance. From the equality $M_O = 0$ it follows that the algebraic sum of the moments of all couples resulting when the given system of forces is reduced to the centre O is zero and so, by the equilibrium condition of couples (see the corollary in Sec. 24), these couples also mutually balance. From this it is clear that to obtain equilibrium for a plane system of forces it suffices to comply with two conditions:

$$\mathbf{R}_{\mathrm{pr}} = \sum \mathbf{P}_k = 0 \quad \text{and} \quad M_O = \sum m_O(\mathbf{P}_k) = 0.$$

Compliance with both these conditions is not only sufficient for equilibrium but also necessary. Indeed, as we have already seen, if the principal vector of a given system of forces is not equal to zero, then the system reduces to a resultant; if the principal vector is equal to zero and the principal moment of the given system is nonzero, then the system reduces to a couple. It is clear that in both cases the body will not be in equilibrium.

Using the conditions of equilibrium established above, it is easy to prove a theorem bearing the name of the French scientist Varignon (1654-1722):

The moment of a force is the algebraic sum of the moments of its vector components acting at a common point on the line of action of the force.

Proof. Suppose some arbitrary plane system of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots, \mathbf{P}_n$ (Fig. 66, see p. 110) has a resultant $\mathbf{R}_P$. We bring this system to equilibrium by adjoining to it a balancing force $\mathbf{R}'$, which, by corollary 2 of the axioms of statics, must be equal in magnitude to the resultant $\mathbf{R}_P$ and opposite in direction along one straight line. Since the system of forces $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots, \mathbf{P}_n)$ is in equilibrium, it must satisfy the above-established conditions of equilibrium, one of which for the given system of forces is that the algebraic sum of the moments of all forces about any point O of the plane of action of the forces be equal to zero. Consequently,

$$\sum m_O(\mathbf{P}_k) + m_O(\mathbf{R}') = 0.$$

But since the forces $\mathbf{R}'$ and $\mathbf{R}_P$ are equal in magnitude and opposite in direction along a single straight line, it follows that the algebraic magnitudes of their moments about any point will also be equal in magnitude and opposite in sign, that is, $m_O(\mathbf{R}') = -m_O(\mathbf{R}_P)$. Substituting this value into the preceding equality, we get

$$\sum m_O(\mathbf{P}_k) - m_O(\mathbf{R}_P) = 0$$

whence

$$m_O(\mathbf{R}_P) = \sum m_O(\mathbf{P}_k). \tag{27}$$

The theorem is proved.

A consequence of the Varignon theorem is the condition of equilibrium of a lever:

A *lever*, in the broad sense, *is a rigid body* (*bar*) *pivoted about a fixed axis and under the action of forces lying in a plane perpendicular to that axis.*

The point of intersection of the lever axis and the plane of action of the forces is termed the fulcrum.

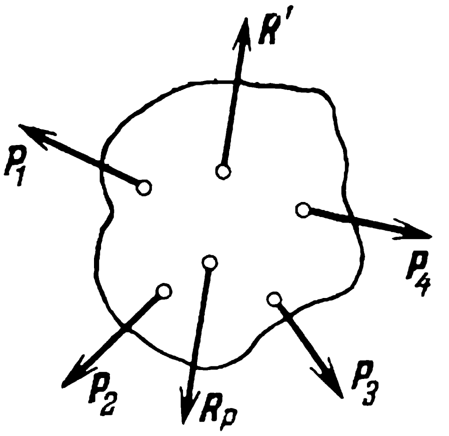


FIG. 66

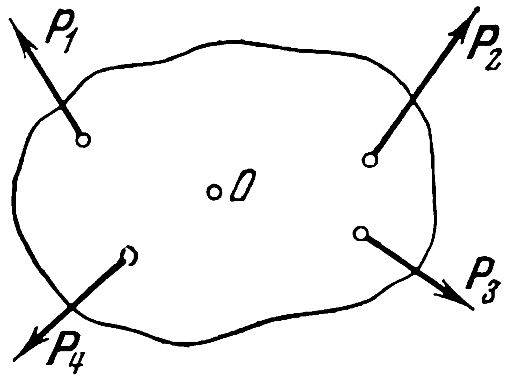


FIG. 67

Suppose (see Fig. 67) a lever is acted upon by a system of forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ lying in the plane of the drawing; let O be the fulcrum.

For equilibrium of the lever it is sufficient that the resultant of all forces applied to the lever (if the resultant exists) pass through the fulcrum, in which case it is balanced by the resistance of the fulcrum. At the same time, this condition is a necessary condition for equilibrium of the lever because if the resultant does not pass through the fulcrum O it imparts a rotation to the lever about that point. But if the resultant passes through the fulcrum, then its moment about this point is $m_O(\mathbf{R}_P) = \sum m_O(\mathbf{P}_k) = 0$.

Now, if the principal vector of the system is zero and the system, consequently, reduces to a couple, then to prevent rotation, the moment of the resultant couple, which is equal to the sum of the moments of the given forces about any point, must also be zero. From this it is clear that *for the equilibrium of a lever it is necessary and sufficient that the algebraic sum of the moments of all forces applied to the lever about the fulcrum be zero.*

Problem 23. A lever AB that can rotate about a fixed axis passing through a point C is in equilibrium under the action of a system of forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ applied as shown in Fig. 68. Determine the magnitude of the force $\mathbf{P}_4$ if $P_1 = 50$ N, $P_2 = 20$ N, $P_3 = 30$ N.

Solution. The magnitude of the force $\mathbf{P}_4$ is determined from the condition of equilibrium of the lever: $\sum m_C(\mathbf{P}_k) = P_1d_1 - P_2d_2 - P_3d_3 - P_4d_4 = 0$. Take point C, the fulcrum, for the centre of the moments.

From Fig. 68 it is seen that $d_1 = AC = 5$ m, $d_2 = DC \times \sin 45° = 3 \times 0.707 = 2.12$ m, $d_3 = CE \times \sin 30° = 2 \times 0.5 = 1$ m, $d_4 = CB \times \sin 60° = 4 \times 0.866 = 3.46$ m. Substituting these values

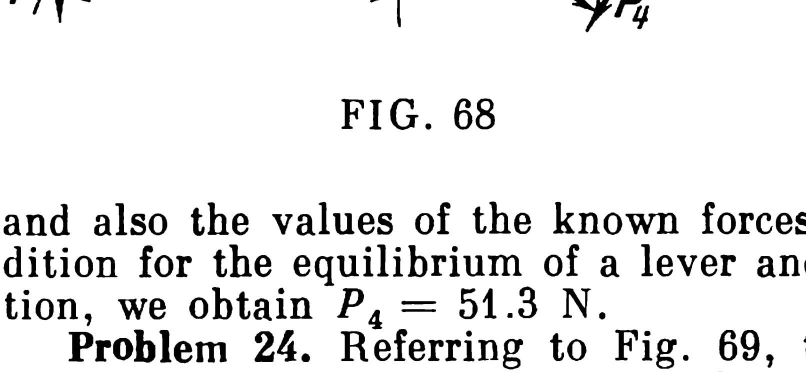


FIG. 68

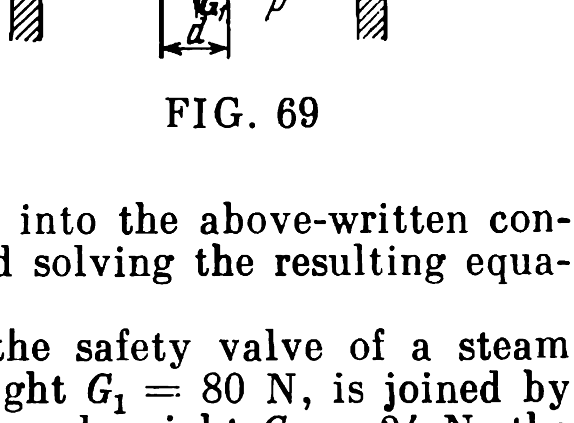


FIG. 69

and also the values of the known forces into the above-written condition for the equilibrium of a lever and solving the resulting equation, we obtain $P_4 = 51.3$ N.

Problem 24. Referring to Fig. 69, the safety valve of a steam generator, diameter $d = 65$ mm and weight $G_1 = 80$ N, is joined by a rod AB to a lever OC of length 60 cm and weight $G_2 = 24$ N, the lever rotates on a hinge O; $OA = 8$ cm. At what point D of the lever OC should a load weighing $G = 250$ N be suspended in order to open the valve when the pressure in the boiler reaches $p = 500$ kPa?

Solution. Applied to the lever OC are parallel forces: at point A the force $\mathbf{P}$ of steam pressure and the force $\mathbf{G}_1$ of gravity (weight of the valve) that act along the same straight line but in opposite directions, at point E (at the midpoint of the lever) the force $\mathbf{G}_2$ of gravity (weight of the lever), and at some point D the force $\mathbf{G}$ of gravity (weight of the load).

The condition of equilibrium of the lever is

$$\sum m_O(\mathbf{P}_k) = P \times OA - G_1 \times OA - G_2 \times OE - G \times OD = 0.$$

The force of pressure of the steam on the valve is

$$P = \frac{\pi d^2}{4} p = \frac{3.14 \times 0.065^2}{4} \times 500 = 1.66 \text{ kN} = 1660 \text{ N}.$$

Substituting the value of the force **P** thus found and also the known values of the arms into the preceding equation, we get $OD = 45.8$ cm.

Sec. 30. Different forms of equations of equilibrium of an arbitrary plane system of forces

As has already been demonstrated (Sec. 29), for the equilibrium of an arbitrary plane system of forces it is necessary and sufficient that its principal vector $\mathbf{R}_{\text{pr}} = \sum \mathbf{P}_k$ and the algebraic magnitude of the principal moment M_O about any point O lying in the plane of action of the forces be equal to zero. On this basis, we can establish three distinct forms of the equilibrium equation for an arbitrary plane system of forces.

The first (basic) form of the equations. From the formulas for the magnitude of the principal vector and the principal moment of a plane system of forces,

$$\mathbf{R}_{\text{pr}} = \sqrt{(\sum X_k)^2 + (\sum Y_k)^2} \quad \text{and} \quad M_O = \sum m_O(\mathbf{P}_k),$$

it follows immediately that $\mathbf{R}_{\text{pr}}$ and M_O are equal to zero when the following equations hold:

$$\sum X_k = 0, \quad \sum Y_k = 0, \quad \sum m_O(\mathbf{P}_k) = 0. \qquad (28)$$

Thus, *for the equilibrium of a plane system of arbitrarily arranged forces, it is necessary and sufficient that the sums of the projections of all forces on each of the two coordinate axes lying in the plane of action of the forces and the algebraic sum of the moments of all forces about any point of that plane be zero.*

This form of the equations is used most frequently when solving problems on the equilibrium of an arbitrary plane system of forces. But in some cases it is more convenient to use one of the other two forms of equilibrium equations.

It is important to point out that in all cases *only three independent equilibrium equations can be set up for an arbitrary plane system of forces.*

The second form of equilibrium equations (the theorem of three moments). *For the equilibrium of a plane system of arbitrarily arranged forces, it is necessary and sufficient that the algebraic sum of the moments of all forces about any three points (not lying on one straight line) of the plane of action of the forces be equal to zero:*

$$\sum m_A(\mathbf{P}_k) = 0, \quad \sum m_B(\mathbf{P}_k) = 0, \quad \sum m_C(\mathbf{P}_k) = 0 \quad (29)$$

if the points A, B and C do not lie on one straight line.

The necessity of these equations follows from the fact that in the case of equilibrium of a plane system of forces, the algebraic sum of the moments of all its forces about any point of the plane must be zero. The sufficiency of these three equations for asserting the equilibrium of a plane system of forces stems from the following reasoning.

The system cannot be reduced to a couple since in that case the principal moment of the system about any point is equal to the moment of the couple and cannot be zero. Neither can the system be reduced to a resultant. By the Varignon theorem, the moment of the resultant about any point is equal to the sum of the moments of the constituent forces about that point. When a resultant exists, its moment about any point may be zero only if the line of action of the resultant passes through that point. But since the three given points A, B and C do not lie on a single straight line, the line of action of the resultant cannot simultaneously pass through these three points.

The third form of the equations of equilibrium. *For the equilibrium of a plane system of arbitrarily located forces, it is necessary and sufficient that the algebraic sum of the moments of all forces about any two points of the plane and the sum of the projections of all forces on any one axis lying in the same plane but not perpendicular to the straight line passing through the chosen centres of the moments be equal to zero:*

$$\sum m_A(\mathbf{P}_k) = 0, \quad \sum m_B(\mathbf{P}_k) = 0, \quad \sum X_k = 0 \quad (30)$$

if the x-axis is not perpendicular to the straight line AB.

The necessity of these conditions stems from the fact that in the case of the equilibrium of a plane system of forces, both the sum of the projections of all forces on any axis and also the sum of the moments of all forces about any point are equal to zero.

Sufficiency, however, requires proof. By the Varignon theorem, it follows from $\sum m_A(\mathbf{P}_k)=0$ and $\sum m_B(\mathbf{P}_k)=0$ that the line of action of the resultant $\mathbf{R}_P$ (if it exists) must definitely pass through the points A and B. But in that case its projection ($R_{Px}=\sum X_k$) on the x-axis that is not perpendicular to AB cannot be equal to zero (Fig. 70). This contradicts the condition $\sum X_k=0$ and therefore the resultant $\mathbf{R}_P$ must, under the given conditions, be zero and the system will be in equilibrium.

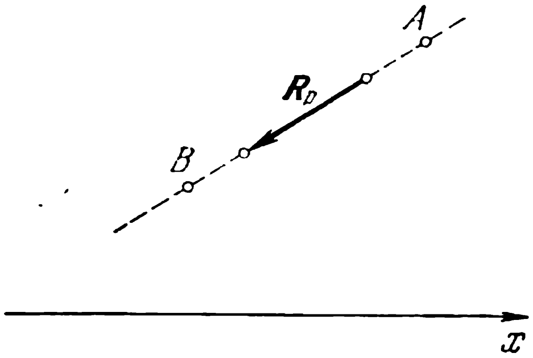


FIG. 70

Sec. 31. Remarks on solving problems involving the equilibrium of a plane system of forces

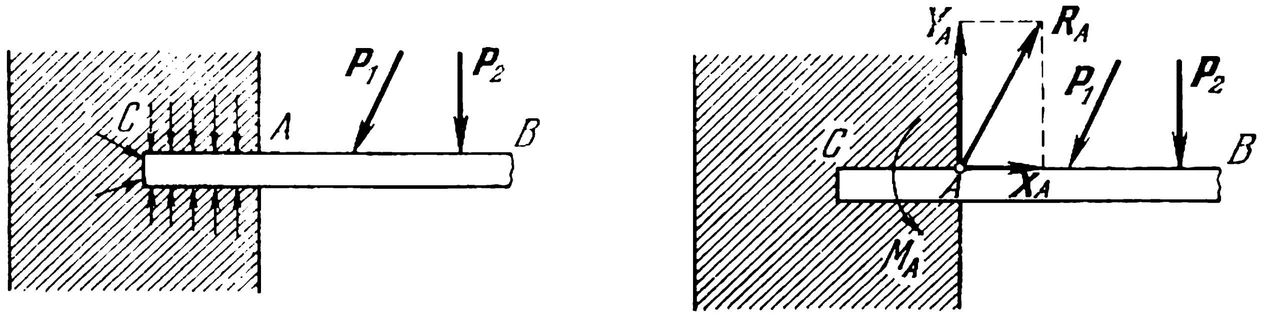
The procedure for solving problems on the equilibrium of a system of forces located arbitrarily on a plane is the same as that for converging forces. Besides what has been said in Sec. 16, one need only recommend that the centre of moments be taken at a point lying on the line of action of one of the unknown forces (still better is the point of intersection of the lines of action of two unknown forces, provided the position of that point can be determined with ease). The moment of the force about the thus chosen centre will be zero (because its arm is zero) and this unknown force will consequently be eliminated from the equation of moments.

All axioms and propositions of statics are established for so-called concentrated forces, that is, for forces applied to certain points of a rigid body. In practice, however, one often has to do with forces distributed along a given surface in accord with some law. In solving problems of statics, such a system of forces has to be replaced by its resultant. How this is done in the simplest case of a plane system of uniformly distributed forces is shown in problems 28 and 30.

If a plane system of forces acting on a body at equilibrium involves a couple (as, for instance, in problems 28 and 30), then one must bear in mind, when setting up the equilibrium equations, that the algebraic sum of the projections of the forces of any couple (which is always a system of two equal-in-magnitude, parallel and oppositely directed forces) on any axis is zero. Now the algebraic sum of the moments of the forces of a couple about any point of a plane is, according to theorem 1 of Sec. 23, equal to the algebraic magnitude of the moment of the couple.

As we already know, many problems of statics consist in determining the reactions of constraints, in particular, in determining the reactions of supports of various kinds of beam systems, trusses, etc.

Besides movable and fixed hinged supports mentioned in Sec. 7, there is also a support of practical interest that is accomplished by means of the rigid fixing of the end of a beam (Fig. 71*a*). This support does not allow for linear displacements of the beam (as in the case of a fixed-hinge support) or for any turning either.



(a) (b)

FIG. 71

Acting on the fixed end of the beam *AC* from the surfaces on which it rests are the nonuniformly distributed reactions of these surfaces. Using the Poinsot theorem, we can reduce them to a single point (Fig. 71*b*) and replace them by a single force—the reaction $\mathbf{R}_A$ applied to point *A* and equal to the principal vector of the distributed reactions, and by a single couple with moment M_A equal to the principal moment of these forces about the point *A*, and called the *moment of end-restraint reaction*. Finding the unknown magnitude and direction of the reaction $\mathbf{R}_A$ may in turn be replaced by finding the algebraic values X_A and Y_A of the two components.

Hence (see problem 28), finding the reaction of a rigidly fixed beam reduces, in the general case of forces arbitrarily located in the plane, to determining three unknown quantities: the algebraic magnitudes of the components X_A and Y_A of the reaction $\mathbf{R}_A$ that prevent any linear displacement of the beam in the plane of action of the forces, and the algebraic magnitude of the moment of end-restraint reaction M_A that prevents rotation of the beam under the action of active forces applied to it.

Problem 25. A homogeneous rod AB (see Fig. 72) of length 3 m and weight $G = 300$ N rests at A on a smooth horizontal floor at an angle of $\alpha = 60°$ and at points C and D on two rollers. Determine the forces of pressure of the rod on the floor and on the rollers if the distances $AC = CD = DB = 1$ m.

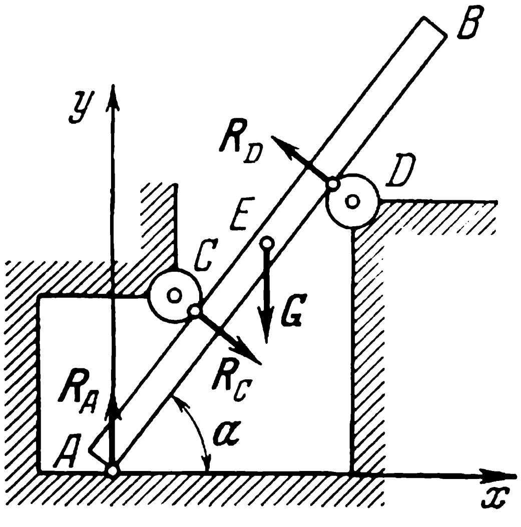


FIG. 72

Solution. The forces of pressure of the rod on the floor and on the rollers are equal in magnitude to the corresponding reactions of the floor and rollers. The reaction of the smooth floor $\mathbf{R}_A$ is perpendicular to the floor. The reactions of the rollers $\mathbf{R}_C$ and $\mathbf{R}_D$ (if the slight friction is disregarded) are perpendicular to the rod surface. Replacing restraints (the supports A, C and D) with corresponding reactions, we may regard the rod as being free and in equilibrium under the action of the forces $\mathbf{R}_A$, $\mathbf{R}_C$, $\mathbf{R}_D$ and $\mathbf{G}$ (the force of gravity of the homogeneous rod applied at its midpoint, point E) lying in the same plane. Drawing the coordinate axes x and y as shown in the drawing, and taking for the centre of moments the point A, we set up a table of the projections of all forces on the chosen coordinate axes and of their moments about A.

Force	Projections of force on axes		Moment of force about A
	x	y	
$\mathbf{G}$	0	$-G$	$-G\times\frac{AB}{2}\times\cos\alpha$
$\mathbf{R}_A$	0	R_A	0
$\mathbf{R}_C$	$R_C\sin\alpha$	$-R_C\cos\alpha$	$-R_C\times AC$
$\mathbf{R}_D$	$-R_D\sin\alpha$	$R_D\cos\alpha$	$R_D\times AD$

The equilibrium equations (28) take the form

$$\sum X_k = R_C \sin\alpha - R_D \sin\alpha = 0$$

whence $R_C = R_D$,

$$\sum Y_k = -G + R_A - R_C \cos\alpha + R_D \cos\alpha$$

whence $R_A = G = 300$ N,

$$\sum m_A(\mathbf{P}_k) = -G \times \frac{AB}{2} \times \cos\alpha - R_C \times AC + R_D \times AD = 0.$$

Substituting the values of **G**, AB, AC, AD and α and solving the last equation, we obtain $R_C = R_D = 225$ N.

Problem 26. With reference to Fig. 73, a car weighing $G = 10$ kN is held on an inclined plane by a rope passed over a pulley and parallel to the plane. The angle α is 30°. Determine the forces of pressure of the wheels of the car on the plane at points A and B and the tension of the rope if the centre of gravity of the car is at point C; $AD = DB = a = 0.75$ m, $CE = b = 0.3$ m.

Solution. Regard the car as being free and in equilibrium under the action of the following forces: the force of gravity **G** of the car applied at its centre of gravity C, the reaction of the rope **T**, and the reactions of the plane $\mathbf{R}_A$ and $\mathbf{R}_B$ perpendicular to the inclined plane. Arrange the coordinate axes as shown in Fig. 73. Take E as the centre of moments. The values of the projections of all forces on the chosen coordinate axes and the moments of these forces about E are tabulated below.

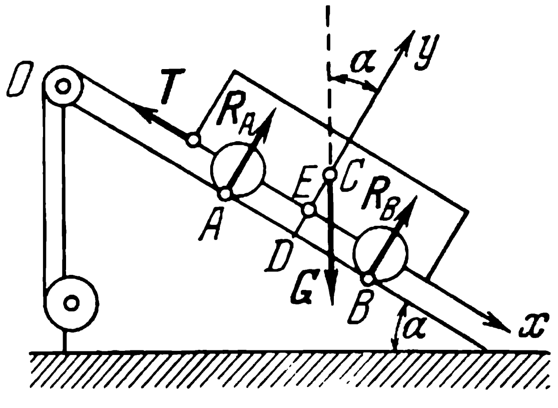


FIG. 73

$\mathbf{P}_k$	X_k	Y_k	$m_E(\mathbf{P}_k)$
T	$-T$	0	0
$\mathbf{R}_A$	0	R_A	$-R_A a$
$\mathbf{R}_B$	0	R_B	$R_B a$
G	$G\cos(90° - \alpha) = G\sin\alpha$	$-G\cos\alpha$	$-Gb\sin\alpha$

The equilibrium equations are:

$$\sum X_k = -T + G \sin \alpha = 0,$$

$$\sum Y_k = R_A + R_B - G \cos \alpha = 0,$$

$$\sum m_E (\mathbf{P}_k) = -R_A a + R_B a - Gb \sin \alpha = 0.$$

Substituting the given values and solving the system of equations, we find

$$T = 5 \text{ kN}, \quad R_A = 5.33 \text{ kN}, \quad R_B = 3.33 \text{ kN}.$$

Problem 27. The axis AB of a hoisting crane weighing $G = 15$ kN rotates in a thrust bearing B and in a bearing A (Fig. 74). A load $Q = 8$ kN is suspended at C. Determine the reactions of the thrust bearing and the bearing if the distance $AB = 4$ m, the distance of the centre of gravity of the crane D from the axis of rotation $KD = 1$ m and the distance of C from that same axis $EC = 2$ m.

Solution. Since the thrust bearing B prevents any translational motion of the crane, its reaction can be decomposed into a horizontal component $\mathbf{X}_B$ and a vertical component $\mathbf{Y}_B$. The bearing A which does not prevent displacement of the crane along its axes only yields a single horizontal reaction $\mathbf{X}_A$ that is perpendicular to the axis of rotation of the crane. We eliminate the constraints and consider the equilibrium of the crane under the action of all forces applied to it (action forces and constraint reactions). Arrange the coordinate axes as shown in the drawing. Point B is conveniently taken for the centre of moments since the lines of action of the unknown forces $\mathbf{X}_B$ and $\mathbf{Y}_B$ pass through it and, consequently, their moments about that point are zero.

Let us set up the equations of equilibrium of the crane:

$$\sum X_k = -X_A + X_B = 0, \quad \sum Y_k = -G - Q + Y_B = 0,$$

$$\sum m_B (\mathbf{P}_k) = -G \times KD - Q \times EC + X_A \times AB = 0.$$

Substituting the data and solving the system of equations, we obtain

$$Y_B = G + Q = 23 \text{ kN}, \quad X_A = 7.75 \text{ kN}, \quad X_B = X_A = 7.75 \text{ kN}.$$

Problem 28. A cantilever beam AB is loaded as shown in Fig. 75. The intensity* of a load q uniformly distributed over the whole length of the beam is $q = 20$ kN/m. Beam length $l = 2$ m. A concentrated load $P = 40$ kN is applied at point B of the beam and is directed downwards at an angle of $\alpha = 30°$ to the vertical. Determine the reactions of rigid restraint and ignore the weight of the beam.

---

* The intensity q of a load is defined as the numerical value of the load per unit length of the loaded portion of the beam.

Solution. We consider the equilibrium of the beam. The action forces applied to the beam are: the uniformly distributed load whose resultant $Q = ql = 20 \times 2 = 40$ kN is applied to the midpoint of the beam, and the concentrated force **P**. A rigid constraint is imposed at point A on the beam (the beam is fixed in the wall); this prevents any linear displacements of the beam and prevents it from turning about point A. As already mentioned (in Sec. 31), the action of such a force is equivalent to the action of a single reaction $\mathbf{R}_A$ and of a certain couple.

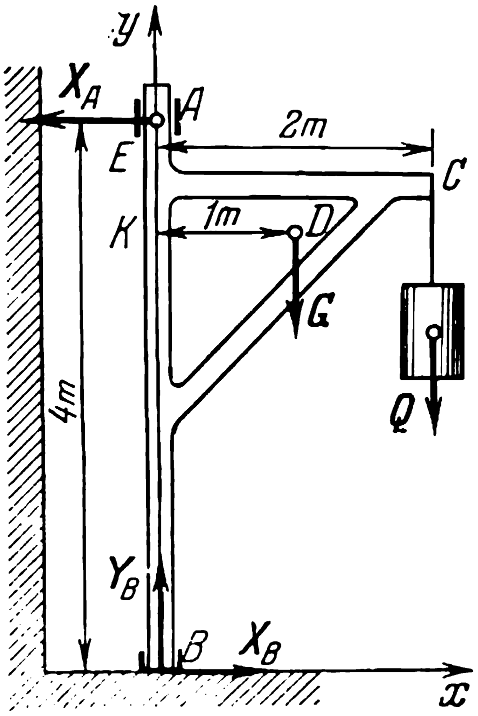


FIG. 74

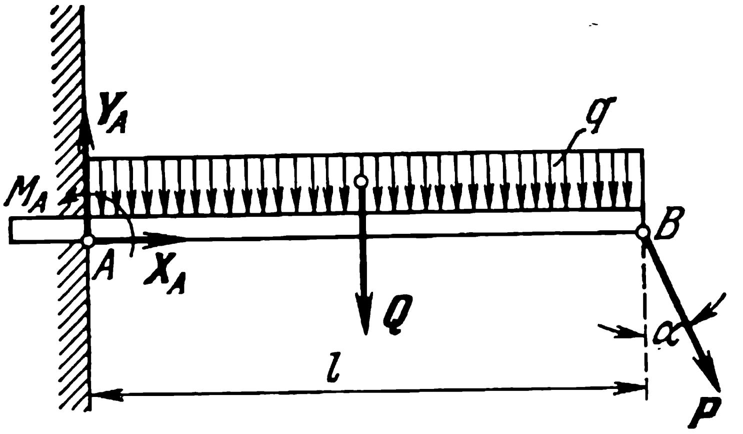


FIG. 75

The sense of rotation of this couple is, generally speaking, as yet unknown. We take the moment M_A of the couple to be positive, that is, we consider that the couple tends to turn the beam counterclockwise.

If when solving the problem the value of this moment turns out to be negative, that will mean that in reality the sense of rotation of the couple is contrary to what was assumed.

Since the direction of the force of reaction $\mathbf{R}_A$ is also, generally speaking, unknown, we decompose it into the components $\mathbf{X}_A$ and $\mathbf{Y}_A$ whose directions are shown in the drawing.

Thus, the given beam is in a state of equilibrium under the action of a plane system of forces **Q**, **P**, $\mathbf{X}_A$, $\mathbf{Y}_A$ and a couple with moment M_A.

We now set up the equations of equilibrium of the beam:

$$\sum X_k = X_A + P \sin \alpha = 0, \qquad \sum Y_k = Y_A - Q - P \cos \alpha = 0,$$

$$\sum m_A (\mathbf{P}_k) = M_A - Q \frac{l}{2} - Pl \cos \alpha = 0.$$

Solving this system of equations and substituting the data, we obtain

$$X_A = -P \sin \alpha = -20 \text{ kN}, \quad Y_A = Q + P \cos \alpha = 74.6 \text{ kN},$$

$$M_A = Q\frac{l}{2} + Pl \cos \alpha = 109.3 \text{ kN} \cdot \text{m}.$$

The negative value obtained for X_A shows that in reality the component $\mathbf{X}_A$ of the reaction is in a direction opposite that which was assumed.

Sec. 32. The equations of equilibrium of a plane system of parallel forces

Since a parallel location of forces on a plane is a special case of their arbitrary arrangement in the plane, the three equations of equilibrium of a plane system of forces that were established in the preceding section can be applied to such a system as well:

$$\sum X_k = 0, \quad \sum Y_k = 0, \quad \sum m_O(\mathbf{P}_k) = 0.$$

Using the fact that the axes of projections may be located arbitrarily in the plane of action of the forces, we draw the y-axis parallel to the given forces, and the x-axis perpendicular to those forces (Fig. 76).

The projection of each one of the forces on the x-axis perpendicular to them is zero, and therefore the first equation becomes the identity $0 = 0$ for all values of the forces. Hence, the equation $\sum X_k = 0$ holds true for a system of parallel forces, irrespective of whether the system is in equilibrium or not. Given such a choice of the axes of projections for parallel forces, the equation $\sum X_k = 0$ ceases to be an equation of equilibrium and therefore is dropped.

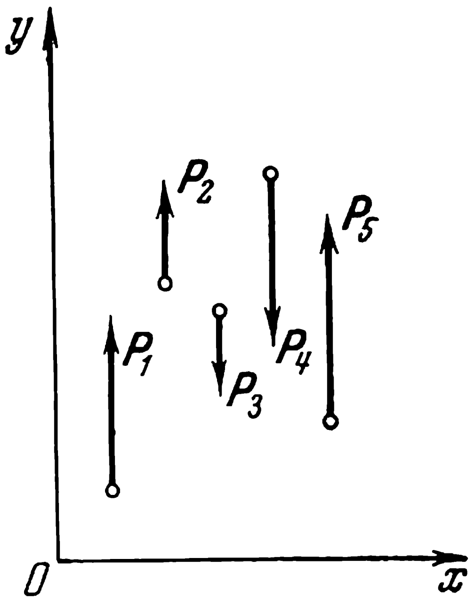


FIG. 76

Since all the given forces are parallel to the y-axis, the projection of each force on that axis is equal to the magnitude of the force taken with the appropriate sign. Hence, $\sum Y_k = \sum(\pm P_k)$. Thus, the equations of equilibrium for a plane system of parallel forces take the form

$$\sum(\pm P_k) = 0, \quad \sum m_O(\mathbf{P}_k) = 0. \tag{31}$$

For the equilibrium of a plane system of parallel forces, it is necessary and sufficient that the algebraic sum of all forces and the algebraic sum of the moments of all forces about any point lying in the plane of action of the forces be equal to zero.

Recalling what was said in Sec. 30 about the third possible form of the equations of equilibrium of a plane system of forces and taking for the axis of projections the axis perpendicular to the parallel forces, the equations of equilibrium of parallel forces may be given a different form, namely:

$$\sum m_A(\mathbf{P}_k) = 0, \quad \sum m_B(\mathbf{P}_k) = 0. \tag{32}$$

For the equilibrium of a plane system of parallel forces, it is necessary and sufficient that the algebraic sums of the moments of all forces about each of two arbitrarily chosen points of the plane, but such that do not lie on a straight line parallel to the given forces, be equal to zero.

Problem 29. Referring to Fig. 77, vertical forces $P_1 = 6$ kN and $P_2 = 4$ kN are applied at points A and D to a horizontal cantilever beam AB having a fixed support B and a movable support C. Find the support reactions if $AC = 1.2$ m, $CD = 0.9$ m and $DB = 2.1$ m.

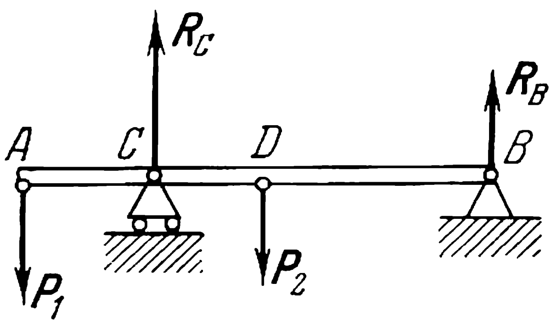


FIG. 77

Solution. The reaction $\mathbf{R}_C$ of the movable support C is perpendicular to the horizontal fixed support surface of rollers and, hence, is parallel to the vertical action forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied to the beam. The direction of the reaction $\mathbf{R}_B$ of the fixed support B is generally speaking indeterminate. But the beam is in a state of equilibrium due to the action forces and constraint reactions applied to it, whereas the system of parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{R}_C$ may be balanced only by a force parallel to them. Thus, in the given case, the reaction of the fixed support B can also only be vertical, and the four forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{R}_C$ and $\mathbf{R}_D$ that are in equilibrium must satisfy the two conditions of equilibrium of parallel forces in the plane.

Let us take the algebraic sum of moments of all forces about some point. The simplest solution is obtained if for the centre of moments we take one of the points of the support, say C:

$$\sum m_C(\mathbf{P}_k) = P_1 \times AC + R_C \times 0 - P_2 \times CD + R_B \times CB = 0,$$

whence

$$R_B = \frac{P_2 \times CD - P_1 \times AC}{CB} = \frac{4 \times 0.9 - 6 \times 1.2}{3} = -1.2 \text{ kN}.$$

The minus sign shows that the reaction of support B is directed downwards and not upwards, as was assumed. The reaction of the second support may be found by equating to zero either the sum of the moments of all forces about another point (most conveniently, point B) or the algebraic sum of all forces $\sum(\pm P_k) = -P_1 + R_C - P_2 + R_B = 0$, whence $R_C = P_1 + P_2 - R_B = 6 + 4 - (-1.2) = 11.2$ kN.

Problem 30. Referring to Fig. 78, we see that a horizontal beam with two overhangs is acted upon by a couple with moment m = 20 kN·m; the force acting on the right-hand overhang is a uniformly distributed load of intensity $q = 75$ kN/m, that acting at point C of the left-hand overhang is a vertical concentrated load $P = 30$ kN. The dimensions (in metres) of the beam are as indicated in Fig. 78. Determine the reactions of the supports A and B.

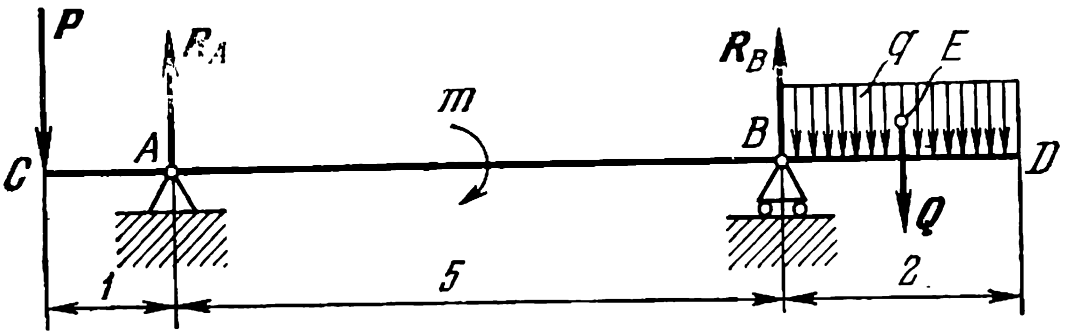


FIG. 78

Solution. To determine the reactions of the supports, replace the distributed load acting on the section BD of length $l = 2$ m by the resultant. Since the load is uniformly distributed throughout the length of the section, its resultant $Q = ql = 7.5 \times 2 = 15$ kN is applied to point E, the midpoint of BD.

The reaction $\mathbf{R}_B$ of the movable support and the action forces $\mathbf{P}$ and $\mathbf{Q}$ applied to the beam are vertical. Since a couple can only turn a body and cannot impart to the beam a horizontal displacement, the reaction $\mathbf{R}_A$ of the fixed support A will also be directed vertically.

We now set up the equations (32) of equilibrium of the beam. Since (see Sec. 23) the sum of the moments of the forces of a couple about any centre is equal to the moment of the couple and the given couple rotates the plane of the drawing clockwise, it follows that

$$\sum m_A(\mathbf{P}_k) = P \times AC - m + R_B \times AB - Q \times AE = 0,$$

whence

$$R_B = \frac{Q \times AE + m - P \times AC}{AB} = \frac{15 \times 6 + 20 - 30 \times 1}{5} = 16 \text{ kN},$$

$$\sum m_B(\mathbf{P}_k) = P \times BC - R_A \times AB - m - Q \times BE = 0,$$

from which

$$R_A = \frac{P \times BC - m - Q \times BE}{AB} = \frac{30 \times 6 - 20 - 15 \times 1}{5} = 29 \text{ kN}.$$

This result may be checked. Since the beam is in equilibrium, the equation $\sum(\pm P_k) = 0$ must become an identity when the values of the forces applied to the beam are substituted into it. Indeed, $\sum(\pm P_k) = -P + R_A + R_B - Q = -30 + 29 + 16 - 15 = 0$.

We do not insert the values of the forces of the couple into this equation because their algebraic sum is always zero.

Sec. 33. The equilibrium of a system of linked bodies

A system of linked rigid bodies is defined as a structure consisting of several rigid bodies freely resting on one another or linked together by some kind of nonrigid constraints (hinges, flexible thread).

Since in the case of equilibrium of a system of bodies each body of the system is in a state of equilibrium, problems involving the equilibrium of a system of linked bodies are ordinarily solved by considering the equilibrium of each body separately.

When considering the equilibrium of any body of a system, we must regard the other, linked, bodies as constraints and replace their action on the body under consideration by the reactions of the constraints. In the process it must be borne in mind at all times that the forces of interaction of two bodies are equal in magnitude and opposite in direction along one straight line.

It is sometimes more advantageous to consider first the equilibrium of the entire system as a whole and then the equilibrium of a separate body. In setting up the equations of equilibrium of the whole system, we consider it, in accord with the principle of rigidity, as an absolutely rigid body and therefore these equations will not include forces with which the separate bodies of the system act upon one another. The sum of the projections on any axis of these forces, which are equal in magnitude and opposite in direction, is zero, like the sum of their moments about any point.

Problem 31. A homogeneous ball of weight $G = 600$ N and radius $r = 15$ cm rests on a smooth vertical wall and on a smooth rod AB that is hinged to the wall at point A (Fig. 79*a*). Ignoring the weight

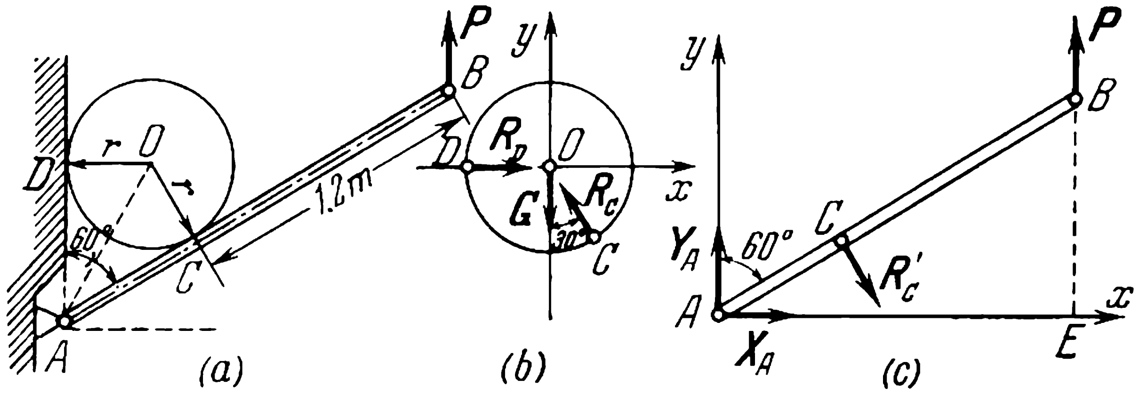


FIG. 79

of the rod AB, determine the vertical force $\mathbf{P}$ that has to be applied to the rod at point B so that the system remains in equilibrium, also determine the reaction of the hinge A and the force of pressure of the ball on the wall. The distance of point C (the point of contact of the ball and rod) to point B (the point of application of force $\mathbf{P}$) is 1.2 m. Angle BAD is 60°.

Solution. We are dealing here with the equilibrium of a system consisting of two bodies: a ball and a rod.

First consider the equilibrium of the ball; replace the constraints imposed on it by their reactions. The ball is in equilibrium under the action of the following forces: the force of gravity $\mathbf{G}$ applied at the geometric centre of the ball, the reaction $\mathbf{R}_C$ of the rod and the reaction $\mathbf{R}_D$ of the wall (Fig. 79*b*); these pass through points C and D (which are points of contact of the ball and are normal to its surface). Thus, the forces applied to the ball converge to its centre.

Choosing the directions of the coordinate axes x and y (Fig. 79*b*), we set up two equations of equilibrium for the given plane system of converging forces.

The reaction $\mathbf{R}_D$ of the smooth vertical wall is perpendicular to it, that is, it is directed horizontally. The rod AB makes an angle of

30° with the horizon and, consequently, the angle between the normal reaction $\mathbf{R}_C$ of that rod and the horizontal x-axis is 60°:

$$\sum X_k = R_D - R_C \cos 60^\circ = 0, \qquad \sum Y_k = R_C \sin 60^\circ - G = 0.$$

Solving this system of equations, we find

$$R_C = \frac{G}{\sin 60^\circ} = \frac{600 \times 2}{\sqrt{3}} = 400\sqrt{3}\ \text{N} \approx 692\ \text{N},$$

$$R_D = R_C \cos 60^\circ = 400\sqrt{3} \times 0.5 = 200\sqrt{3}\ \text{N} \approx 346\ \text{N}.$$

In order to determine the force **P**, applied to rod *AB*, and the reaction of the hinge *A*, we consider the equilibrium of the rod *AB*. Applied to it (Fig. 79*c*) are: the desired vertical force **P**, the force of pressure $\mathbf{R}'_C$ of the ball on the rod, which is equal in magnitude and opposite in direction to the reaction $\mathbf{R}_C$ found above of the rod on the ball, and the unknown (in magnitude and direction) reaction of the hinge *A*. Let us decompose this latter force into two components: a horizontal component $\mathbf{X}_A$ and a vertical component $\mathbf{Y}_A$. Choosing the axes of coordinates and taking for the centre of moments the point *A*, we find the projections of the given plane system of arbitrarily located forces on the coordinate axes and their moments about point *A*.

$\mathbf{P}_k$	X_k	Y_k	$m_A(\mathbf{P}_k)$
$\mathbf{P}$	0	P	$P \times AE$
$\mathbf{R}'_C$	$R'_C \cos 60^\circ$	$-R'_C \sin 60^\circ$	$-R'_C \times AC$
$\mathbf{X}_A$	X_A	0	0
$\mathbf{Y}_A$	0	Y_A	0

The right triangles *ODA* and *OCA* (Fig. 79*a*) are congruent and, consequently,

$$\angle OAC = \frac{\angle DAC}{2} = 30^\circ.$$

From the triangle *OAC* we find the arm *AC*:

$$AC = r \cot 30^\circ = 15\sqrt{3} \approx 26\ \text{cm},$$
$$AB = AC + CB = 26 + 120 = 146\ \text{cm}.$$

From the drawing (Fig. 79*c*) we find the arm *AE*:

$$AE = AB \times \cos 30^\circ = 146 \times \frac{\sqrt{3}}{2} \approx 126.6\ \text{cm}.$$

Setting up the equations of equilibrium and solving them, we find

$$\sum m_A(\mathbf{P}_k) = P \times AE - R'_C \times AC = 0,$$

$$\sum Y_k = P - R'_C \sin 60^\circ + Y_A = 0, \quad \sum X_k = R'_C \cos 60^\circ + X_A = 0,$$

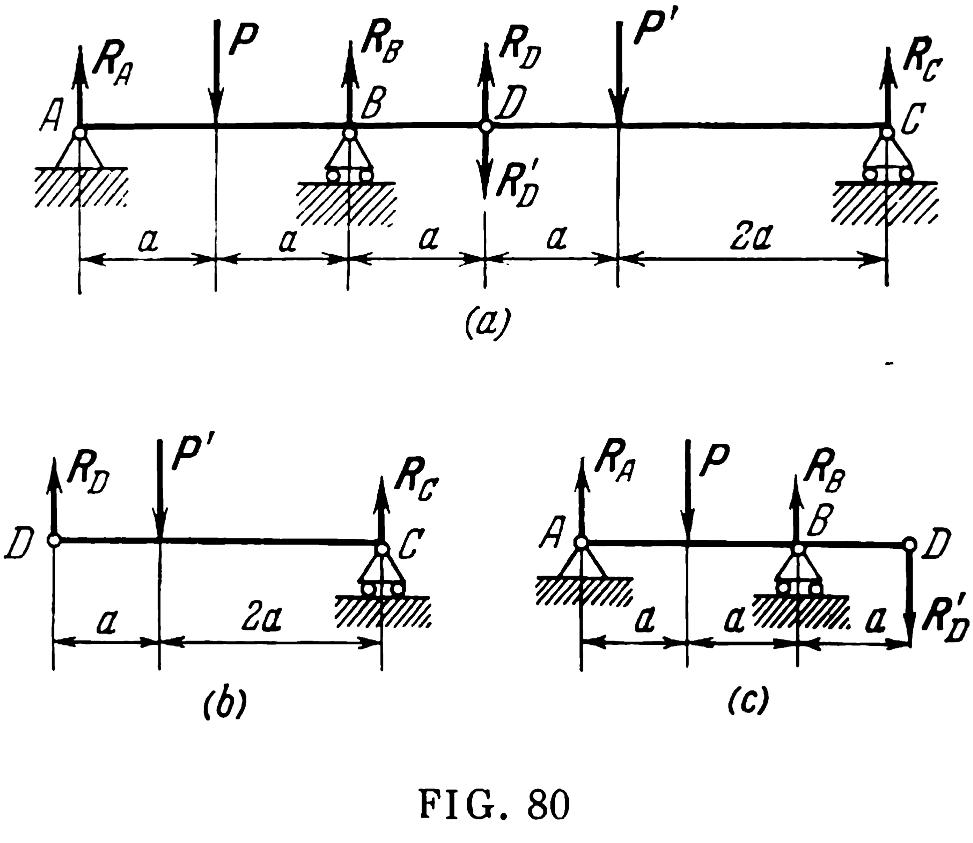
$$P = \frac{R'_C \times AC}{AE} = \frac{692 \times 26.2}{127} \approx 143 \text{ N},$$

$$X_A = -R'_C \cos 60^\circ = -692 \times 0.5 = -346 \text{ N},$$

$$Y_A = R'_C \sin 60^\circ - P = 400\sqrt{3} \times \frac{\sqrt{3}}{2} - 143 = 457 \text{ N}.$$

The negative value of the reaction $\mathbf{X}_A$ shows that it is indeed in the opposite direction to that assumed, that is, directed leftwards.

Problem 32. Find the reactions of the supports A, B and C of a three-support horizontal beam with an intermediate hinge D (Fig. 80).



(a)

(b) (c)

FIG. 80

The location of the supports and vertical loads on the beam is indicated in the drawing (the vertical loads are equal in magnitude).

Solution. The beam AC may be considered as composed of two beams: a cantilever beam AD and a suspended beam DC. The action of one beam on the other (this action is conveyed via the linking hinge D) may be replaced by the reactions of the hinge: $\mathbf{R}_D$ and $\mathbf{R}'_D$. By the law of equality of action and reaction, the force $\mathbf{R}_D$ with which the cantilever beam acts on the suspended beam is equal in magnitude and opposite in direction to the force $\mathbf{R}'_D$ with which the suspended beam acts on the cantilever beam. These reactions, like the reactions of all supports, may be only vertical for the given arrangement of the supports and the directions of the action forces **P** and **P**$'$.

From the conditions of equilibrium of the beam *DC* (Fig. 80*b*), we find

$$\sum m_D(\mathbf{P}_k) = -Pa + R_C \times 3a = 0,$$

whence $R_C = \frac{1}{3}P$,

$$\sum m_C(\mathbf{P}_k) = -R_D \times 3a + P \times 2a = 0,$$

from which

$$R_D = \frac{2}{3}P.$$

The force of pressure of the suspended beam on the cantilever beam is

$$R'_D = R_D = \frac{2}{3}P.$$

From the conditions of equilibrium of the cantilever beam *AD* (Fig. 80*c*), we can find the reactions of the other two supports:

$$\sum m_A(\mathbf{P}_k) = -P \times a + R_B \times 2a - R'_D \times 3a = 0,$$

whence

$$R_B = \frac{P + 3R'_D}{2} = \frac{P + 3 \times \frac{2}{3}P}{2} = \frac{3}{2}P,$$

$$\sum P_k = R_A - P + R_B - R'_D = 0,$$

from which

$$R_A = P + R'_D - R_B = P + \frac{2}{3}P - \frac{3}{2}P = \frac{1}{6}P.$$

Sec. *34. Determining forces in the bars of a truss by the method of cuts

Suppose we have a truss (Fig. 81*a*) that is in equilibrium under the action of external forces applied to it: the action forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and the reactions of the supports $\mathbf{R}_A$ and $\mathbf{R}_B$.

Mentally cut the truss (by means of some section *mn*) into two portions and discard one of them, say the right-hand one.

So that the equilibrium of the remaining portion of the truss is not upset, we must, in accordance with the principle of removability, replace the action of the earlier existing constraints by their reactions, that is, the reactions S_6, S_8 and S_7 of the cut bars on the joints *D* and *F* (Fig. 81*b*).

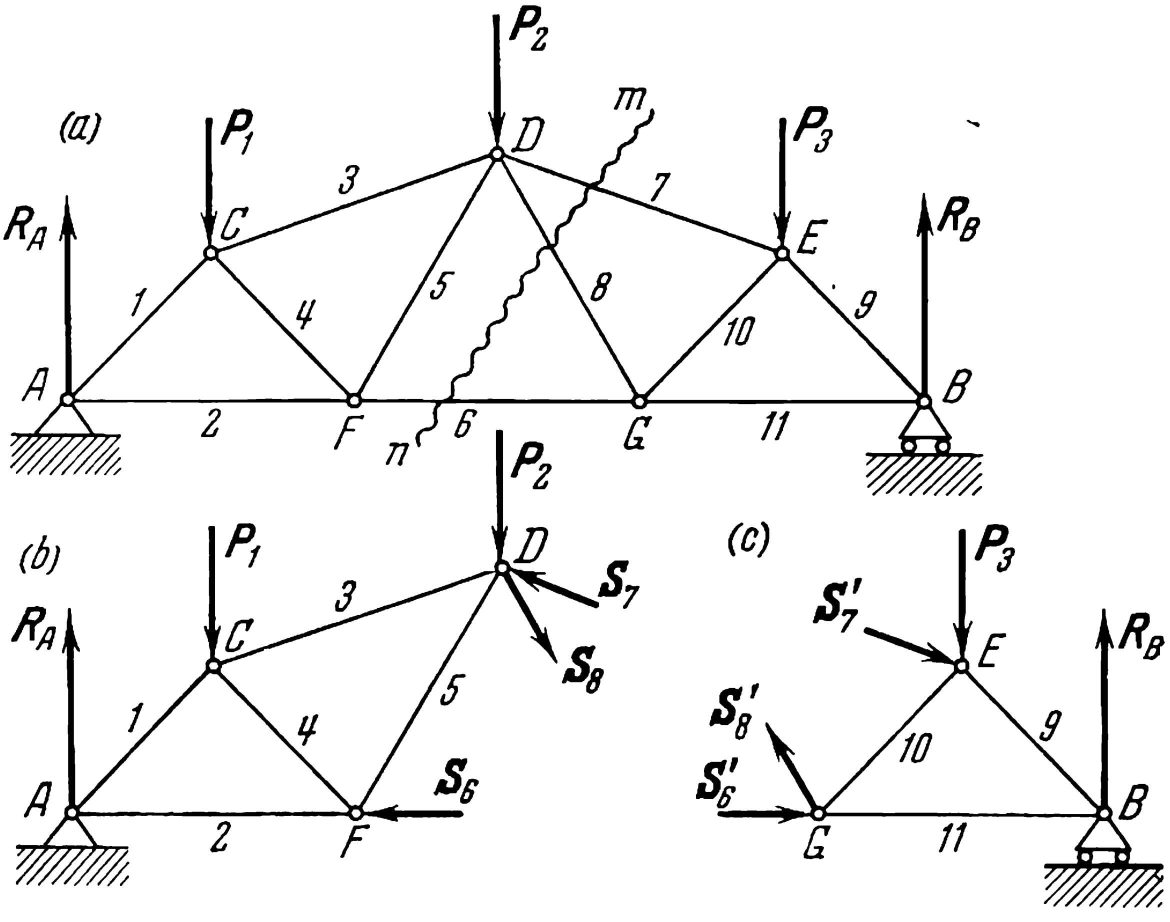


FIG. 81

With just as much justification we could of course have discarded the left-hand portion of the truss and replaced its action on the remaining right-hand portion of the truss by the reactions S'_6, S'_8 and S'_7 of the same bars but this time on the joints *E* and *G* (Fig. 81*c*) and, hence, by the opposite reactions S_6, S_8 and S_7 of these bars on the joints *D* and *F*.

Thus, either of the isolated portions of the truss may be regarded as being in equilibrium under the action of external forces applied to that portion and under the action of the reactions of the cut bars on the appropriate joints of the truss. Clearly, for the purpose of greater simplicity in solving the problem it is more convenient to take that portion of the truss acted upon by a smaller number of reactions.

Setting up the equations of equilibrium for a plane system of forces acting on the isolated portion of the truss, we make it possible to find the reactions of the appropriate ("cut") bars.

The advantage of the given method over that of the isolation of joints is that it permits determining immediately the reaction of any bar of the truss without first determining the reactions of the other bars converging to the preceding joints.

Since only three independent equations of equilibrium can be set up for a plane system of forces, the truss must be sectioned so that not more than three bars with unknown reactions are cut in the process.

The reaction of a bar of the truss may be directed only along the bar away from the joint if the bar is under tension, and towards the joint if it is under compression. To obtain the true sign of force, we will preliminarily assume all bars to be under tension, that is, we will direct their reactions away from the joint. The minus sign in front of the numerical value of the reaction thus found will indicate that its true direction is opposite to that taken, that is, it indicates the bar is under compression. To determine the desired reactions of the bars it is most convenient to set up the equations of equilibrium in the form of equations of moments, taking in succession for the centres of the moments the points at which the lines of action of two of the three unknown reactions intersect.* When it is difficult to calculate analytically the arms of the forces with respect to the centres of the moments, they may be found graphically by drawing a scheme of the portion of the truss under consideration to a definite scale.

If two of the three bars with reactions that have to be determined are parallel and, hence, their point of intersection lies at infinity, we can set up an equation of the projections of the forces on an axis perpendicular to the parallel bars instead of one of the equations of the moments.

Problem 33. Given a bridge truss as depicted in Fig. 82*a* that is acted upon by given vertical forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$, where $P_1 = P_2 =$

---

* The method of cuts of a truss is therefore sometimes called the method of moment points.

$= P_3 = P$. It is required to find the stresses in the bars *10*, *11* and *12* of the truss using the method of cuts.

Solution. First of all, as is always the case in truss analysis, we determine the reactions of the supports. In the given case, they are directed vertically upwards and are equal in magnitude:

$$R_A = R_B = \frac{3}{2} P.$$

Draw a section of the truss through the bars *10*, *11* and *12*, whose reactions it is required to determine, and then discard the left-hand portion of the truss.

In order to keep the right-hand portion of the truss in equilibrium, it is necessary to adjoin the reactions $\mathbf{S}_{10}$, $\mathbf{S}_{11}$ and $\mathbf{S}_{12}$ of the severed bars to the joints K and F. Assuming the bars to be under tension for the time being, direct the reactions as shown in Fig. 82*b*. Set up the equations of equilibrium for the system of forces $\mathbf{P}_3$, $\mathbf{R}_B$, $\mathbf{S}_{10}$, $\mathbf{S}_{11}$ and $\mathbf{S}_{12}$ applied to the isolated portion of the truss.

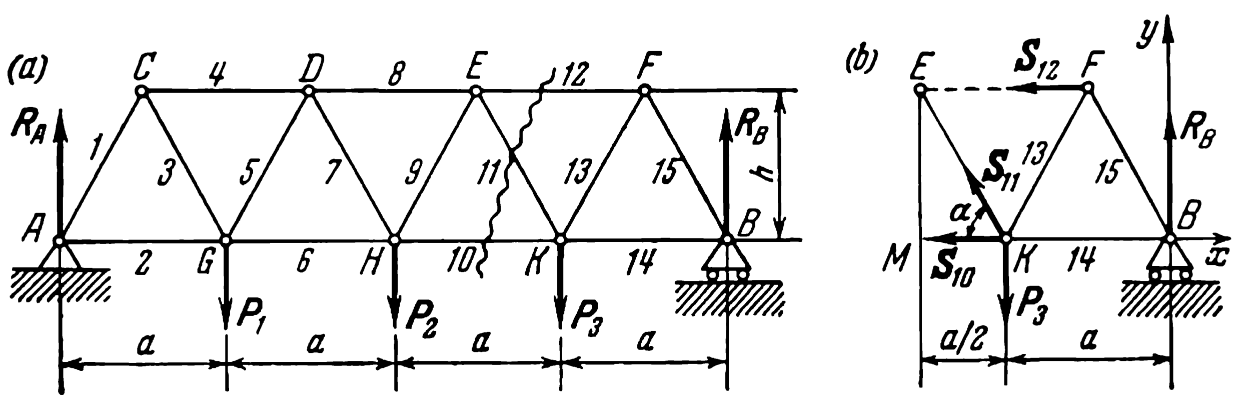


FIG. 82

Now take the sum of the moments of all forces about the point K at which the unknown reactions $\mathbf{S}_{10}$ and $\mathbf{S}_{11}$ converge:

$$\sum m_k (\mathbf{P}_k) = S_{12} h + R_B a = 0,$$

whence

$$S_{12} = -\frac{R_B a}{h} = -\frac{3a}{2h} P.$$

The negative value of the quantity thus found, S_{12}, shows that the reaction is in the opposite direction to that assumed, that is, that the bar *12* is under compression.

We now take the sum of the moments of all forces about point E, at which the unknown reactions $\mathbf{S}_{11}$ and $\mathbf{S}_{12}$ converge:

$$\sum m_E (\mathbf{P}_k) = R_B \frac{3}{2} a - S_{10} h - P_3 \frac{a}{2} = 0,$$

whence

$$S_{10}=\frac{R_B 3a-P_3 a}{2h}=\frac{7a}{4h}P.$$

The bar *10* is under tension.

Since the bars *10* and *12* are parallel and it is impossible to set up an equation of moments in which the reactions of these bars are eliminated, one can take, for the third equation of equilibrium, the equation of the projections on the y-axis (Fig. 82*b*) that is perpendicular to the parallel bars:

$$\sum Y_k=R_B-P_3+S_{11}\sin\alpha=0,$$

whence

$$S_{11}=\frac{P_3-R_B}{\sin\alpha}=-\frac{P}{2\sin\alpha}.$$

The minus sign shows that the bar *11* is under compression.

The angle α is readily found from the right triangle EKM with legs $EM=h$ and $MK=a/2$. Clearly,

$$\tan\alpha=\frac{EM}{MK}=\frac{2h}{a}.$$

Sec. *35. Statically determinate and statically indeterminate problems

The equations of equilibrium established for each instance of force location permit setting up for each case only a definite number of independent equations that impose appropriate conditions on the system of forces that are in equilibrium.

Of course, by projecting the forces on distinct axes and setting up equations of moments about different centres, we can write any number of equations, but there will be only three independent equations for the general case of a plane system (when the forces are located at random in the plane) and only two for the special cases of a plane system: for converging forces and parallel forces in the plane.

This can be illustrated by an example. It is required to determine the reactions of the supports A, B and C of an uncut horizontal beam (Fig. 83) resting on three supports. The arrangement of the supports and the vertical loads

$\mathbf{P}_1$ and $\mathbf{P}_2$ on the beam are indicated in the drawing; the loads are equal in magnitude: $P_1 = P_2 = P$. Reasoning as in the solution of problems 29 and 30, we conclude that the reactions $\mathbf{R}_A$, $\mathbf{R}_B$ and $\mathbf{R}_C$ of all three supports are directed vertically. Thus the beam is in equilibrium under the action of the parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{R}_A$, $\mathbf{R}_B$ and $\mathbf{R}_C$. As we know, one of the conditions of equilibrium of a plane system of parallel forces can be the equation

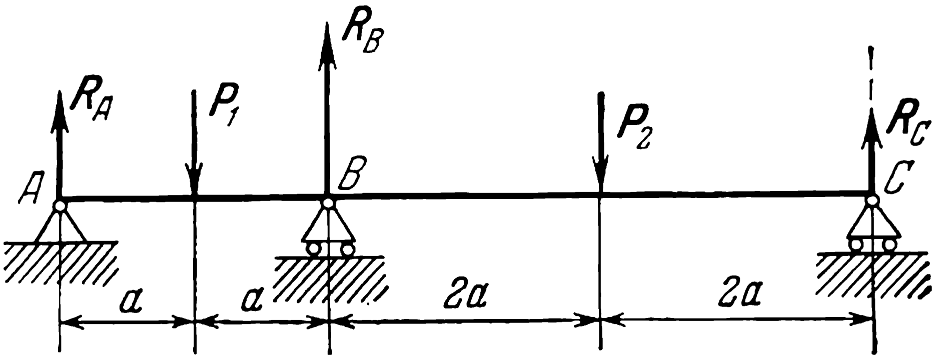


FIG. 83

$$\sum(\pm P_k) = R_A - P + R_B - P + R_C = 0$$

or

$$R_A + R_B + R_C = 2P. \tag{I}$$

For the second condition of equilibrium of parallel forces we can take the zero sum of the moments of all forces about an arbitrary centre lying in the plane of action of the given forces. Taking the points of support for the centres of the moments, we can write:

$$\sum m_A(\mathbf{P}_k) = -Pa + R_B \times 2a - P \times 4a + R_C \times 6a = 0,$$

$$\text{or} \quad 2R_B + 6R_C = 5P, \tag{II}$$

$$\sum m_B(\mathbf{P}_k) = -R_A \times 2a + Pa - P \times 2a + R_C \times 4a = 0,$$

$$\text{or} \quad -2R_A + 4R_C = P, \tag{III}$$

$$\sum m_C(\mathbf{P}_k) = -R_A \times 6a + P \times 5a - R_B \times 4a + P \times 2a = 0,$$

$$\text{or} \quad 6R_A + 4R_B = 7P. \tag{IV}$$

However, despite the apparent external difference of these equations and equation (I), only any two of the four are independent. Indeed, subtracting equation (III) from equation (II) and dividing through by 2, we arrive at equation (I). Adding equations (II) and (IV) and dividing through by 6, we again get equation (I). Similarly, by combining (III) and (IV) and dividing through by 4, we again get equation (I).

In this problem, the magnitudes of the three reactions of the supports are unknown. Hence, the number of unknowns in the problem exceeds the number of independent equations of equilibrium given by the statics of an absolutely rigid body; these equations alone do not suffice to solve the problem.

Problems in which the number of unknown quantities does not exceed the number of independent equations of equilibrium given by the statics of a rigid body for the given arrangement of forces are called *statically determinate problems*, otherwise they are termed *statically indeterminate problems*.

Included in the category of statically indeterminable beams are not only continuous beams lying on three or more supports, but also, for instance, a beam resting on two fixed-hinge supports and loaded with forces that are not perpendicular to the axis of the beam (Fig. 84). There are four unknowns determining the reactions of such a beam, while there are only three equations of statics for forces arbitrarily located in the plane.

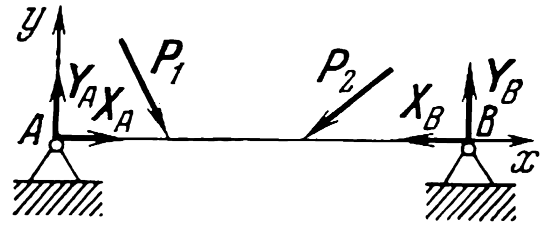


FIG. 84

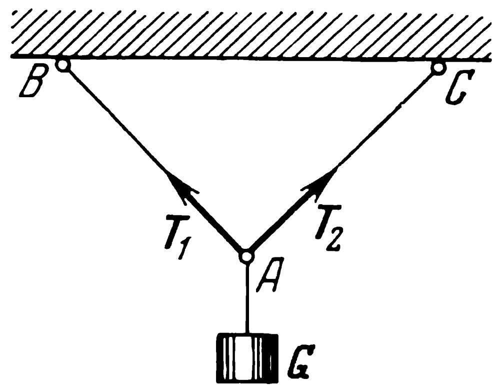


FIG. 85

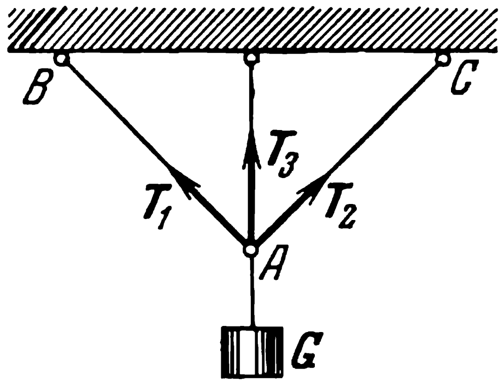


FIG. 86

If a load (Fig. 85) is suspended on two strings AB and AC, we can find the reactions T_1 and T_2 of these strings by regarding the equilibrium of point *A*. The three forces **G**, $\mathbf{T}_1$ and $\mathbf{T}_2$ of which only $\mathbf{T}_1$ and $\mathbf{T}_2$ are unknown, must satisfy two equations of equilibrium of the plane system of converging forces. Thus the number of unknowns (two) is equal to the number of equations (two) and the problem is statically determinate.

Now if the load is suspended on three strings located in one plane (Fig. 86), the point *A* will be in equilibrium under the action of four converging forces: **G**, $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$. Here there are three unknown (in magnitude) forces, whereas we can only set up two equations for the plane system of converging forces. Thus the number of unknowns exceeds the number of equations of statics and the given problem is statically indeterminate.

Methods for solving statically indeterminate problems are considered in strength of materials courses, which are based on theoretical mechanics and make a study of the relationship between deformations of bodies and the forces acting on them. Account of such deformations permits setting up as many supplementary equations as are needed to make the total number of equations (together with the equations of statics of an absolutely rigid body) equal to the number of unknowns in the problem.

CHAPTER VI

FRICTION

Sec. 36. Two basic types of friction

Friction is the resistance to the movement of one body over the surface of another.

Depending on the nature of such movement (whether the body rolls or slides), we distinguish two kinds of friction: *sliding friction* (or *friction of the first kind*) and *rolling friction* (or *friction of the second kind*).*

Examples of sliding friction are: the friction of the runners of a sled on snow, of a saw on wood, of the soles of shoes on the ground, of the bushing of a wheel on the axle. Examples of rolling friction are: the friction of a rolling automobile wheel on the road surface or the wheel of a railway car on the rails, the friction of rolling round logs, the friction in roller bearings and ball bearings.

Friction is one of the most common phenomena of nature and plays a very important role in engineering. However, due to the extreme complexity of this physico-mechanical phenomenon and the difficulties of estimating the numerous factors involved, exact general laws of friction still elude researchers. In practical cases where extreme precision is not required, use is still made of the empirical laws established at the end of the 18th century (1781) by the French scientist Charles Coulomb (1736-1806), although they are only a rough approximation to reality.

If greater precision is required, then one has to determine the magnitude of friction from experiment for each given pair of surfaces involved and for the specific conditions of friction.

---

* At times, yet another type of friction, a pivoting friction, has to be taken into account.

In theoretical mechanics we agreed to consider the motion of imaginary absolutely rigid bodies and not the actual bodies of physical reality. But it is impossible to determine the friction between bodies without an investigation of their physical properties. For this reason, the study of friction lies, strictly speaking, outside the limits of theoretical mechanics. However, friction is frequently touched on in courses of theoretical mechanics, and this is done for the sole reason that certain practical problems do not allow friction to be disregarded.

Sec. 37. Sliding friction

Sliding friction is the resistance generated by the rubbing of bodies in sliding contact.

The basic reason for such friction is that the surfaces of bodies in contact are not absolutely smooth, but are more or less rough. As a result, the sliding of one body over another requires a certain force to overcome the resistance of microscopic irregularities of the surfaces. A certain force is also necessary to overcome the molecular interaction between particles of the surface layers of bodies in contact. To one degree or another, the force of friction arises between all real surfaces, no matter how smooth they are.

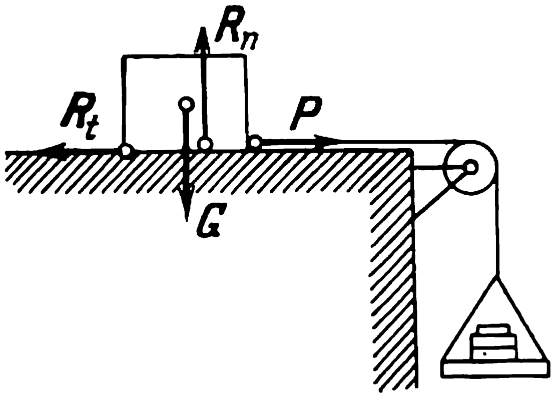


FIG. 87

Referring to Fig. 87, put a block of weight G on a fixed horizontal plane and act on it with a horizontal force **P** by means of a string attached close to the base of the block (to prevent overturning it) and passed across a pulley suspending a plate with weights. The block will remain at rest until the magnitude of the force **P** reaches a certain limit which is definite for the given pair of rubbing surfaces and for a certain pressure between them. What this means is that besides the normal reaction $\mathbf{R}_n$ of the plane, which is equal in magnitude to the force of gravity G, there is another reaction $\mathbf{R}_t$ acting on it due to the plane; this force is equal in magnitude and opposite in direction to the horizontal force **P**. Clearly, it is this reaction $\mathbf{R}_t$ (which lies in the tangential plane) that is the force of friction arising between the surface of the block and the support surface. Just as increasing the weight G of the block increases the magnitude of the normal reaction $\mathbf{R}_n$ of the plane, so increasing the magnitude of the force **P** up to a certain limit (during which the block remains in equilibrium)* increases the magnitude of the frictional force $\mathbf{R}_t$. This force reaches its maximal value when the block begins to move as the force **P** reaches a certain value.

The force of friction that appears when a body is at relative rest is called static friction; *the force of friction that acts when a body is sliding is called the friction of motion* (or *kinetic friction*).

On the basis of numerous experiments, Coulomb established the following (approximate) laws.

1. *All other conditions being equal, the force of friction is independent of the dimensions of the rubbing surfaces.*

For example, if in the preceding experiment we take a block in the form of a parallelepiped, all faces of the same roughness but different surface areas, then the magnitude of the force **P** that must be applied to the block to make it move will not depend on what face it is lying on. It must be noted that this law holds approximately only up to a certain magnitude of the specific pressure, that is, the pressure per unit area of rubbing surfaces. For instance, if the block is put on an edge, the frictional force will be considerably greater.

2. Like the magnitude of any reaction, *the magnitude of the force of static friction depends on the applied forces and, up to a certain limit, is always such as to prevent sliding of one body along the other. However, this cannot exceed a very definite (for each given case) maximal value.*

---

* Since the equal (in magnitude) forces **P** and $\mathbf{R}_t$ do not lie on a single straight line, they form a couple. The moment of this couple is balanced by the couple (G, $\mathbf{R}_n$). Due to the redistribution of pressure of the block on the support plane, the line of action of the normal reaction $\mathbf{R}_n$ of the plane is displaced, and neither will it coincide with the line of action of the force G (of gravity) of the block (Fig. 87).

3. *The maximal value of the force of friction is directly proportional to the force of normal pressure of one body on the other.*

The force of normal pressure is to be understood as the force of pressure directed along the normal to the sliding surface.

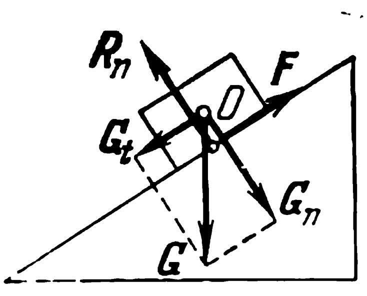


FIG. 88

The force of normal pressure is equal to the weight of the body only if the sliding surface is horizontal and no forces other than the force of gravity are acting. If the body lies on an inclined plane (Fig. 88), then the magnitude of the force of friction is influenced not by the weight but merely by the component $\mathbf{G}_n$ of the force of gravity, which force is perpendicular to the plane and is equal in magnitude to its normal reaction $\mathbf{R}_n$.

If other forces (Fig. 90) are acting on a body besides the force of gravity, then the force of the normal pressure of the body on the surface is to be understood as the normal component of the resultant of all forces applied to the body, which component is equal in magnitude to the normal reaction $\mathbf{R}_n$ of the sliding surface.

4. *The maximal value of the force of friction depends both on the material and the state of the rubbing surfaces and also on the presence and type of lubrication between them.*

For example, the friction of metal on metal is less than the friction of wood on wood, the friction of steel and bronze is less than the friction of steel on steel and so forth. Friction is less if the rubbing surfaces are smooth; that is why the surfaces of contact of rubbing parts of machines are ordinarily polished. Lubricating rubbing surfaces greatly reduces friction. The lubrication fills all irregularities of the rubbing surfaces and lies in a thin layer between them; in this way, direct friction of the surfaces is supplanted by their sliding along the lubricating fluid and the sliding of the separate layers of that fluid along each other.

If we denote by F the maximal value of the static friction and by $\mathbf{R}_n$ the normal reaction of the support surface (this reaction is equal in magnitude to the force of the normal pressure of the body on the support surface), then on the basis of the law we have

$$F = fR_n \tag{33}$$

where f is the coefficient of proportionality called the *coefficient of sliding friction at rest.*

From the equation $f = F/R_n$ it follows that the coefficient of sliding friction is a nondimensional number.

5. *The force of friction in the case of motion is less than the force of friction when the body is at rest.*

Experience shows that to set a body in motion from rest it is necessary (all other conditions being equal) to overcome a greater force of friction than during the motion of the body. The latest experiments have demonstrated that the force of friction in the case of motion depends on the rate of motion of one body relative to the other and in most cases diminishes with increasing speeds and tends to a certain limit.

Materials in contact	Coefficient of friction			
	static		kinetic	
	dry	lubrication	dry	lubrication
Steel on steel	0.15	0.1-0.12	0.15	0.05-0.1
Steel on soft steel	—	—	0.2	0.1-0.2
Steel on iron	0.3	—	0.18	0.05-0.15
Soft steel on iron	0.2	—	0.18	0.05-0.15
Steel on bronze	0.15	0.1-0.15	0.15	0.1-0.15
Soft steel on bronze	0.2	—	0.18	0.07-0.15
Iron on iron	—	0.18	0.15	0.07-0.12
Iron on bronze	—	—	0.15-0.2	0.07-0.15
Bronze on bronze	—	0.1	0.2	0.07-0.1
Soft steel on oak	0.6	0.12	0.4-0.6	0.1
Soft steel on elm	—	—	0.25	—
Iron on oak	0.65	—	0.3-0.5	0.2
Iron on elm or poplar	—	—	0.4	0.1
Bronze on oak	0.6	—	0.3	—
Wood on wood	0.4-0.6	0.1	0.2-0.5	0.07-0.15
Leather (face side) on oak	0.6	—	0.3-0.5	—
Leather (flesh side) on oak	0.4	—	0.3-0.4	—
Leather on iron	0.3-0.5	0.15	0.6	0.15
Rubber on iron	—	—	0.8	0.5

The magnitude of the frictional force in the case of motion can be found from a formula similar to (33) by substituting into that formula the coefficient of friction for motion, f', instead of the coefficient of friction for rest, f.

The above table gives approximate mean values of the coefficients of sliding friction.

In rough calculations, one often dispenses with the difference between the coefficients of static friction and kinetic friction and takes the value of the coefficient of kinetic friction. In precise calculations, the force of friction is determined by experiment for each given pair of rubbing surfaces and for the concrete conditions of friction.

Problem 34. Referring to Fig. 89, a couple with moment $m = 800$ N·m is applied to a shaft. A braking wheel of diameter $d = 40$ cm is jammed on the shaft. Determine the coefficient of static friction between the wheel and the brake shoes if the shoes are pressed to the wheel with forces $Q = Q' = 10$ kN.

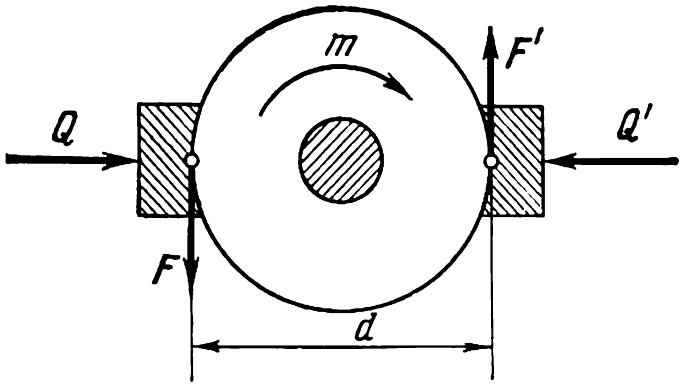


FIG. 89

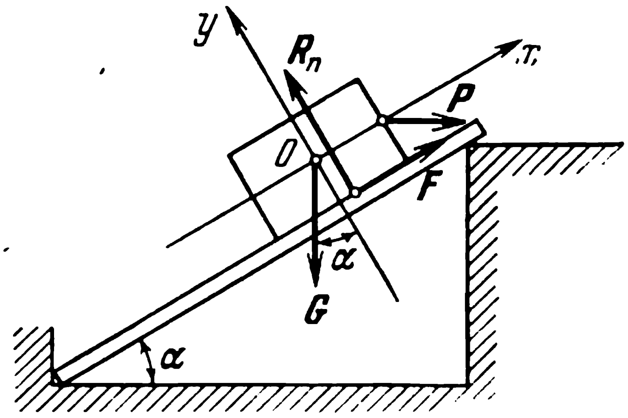


FIG. 90

Solution. The wheel will remain at rest if the moment m of the couple applied to the shaft is balanced by the moment of the couple ($\mathbf{F}$, $\mathbf{F}'$) of the forces of friction between the wheel and the shoes. Hence the condition of equilibrium is $Fd = m$.

The maximal force of static friction between the wheel and the shoes is

$$F = F' = fQ$$

where $Q = Q'$ are the forces of normal pressure of the shoes on the wheel. Substituting the value of the force F into the first equation, we have $fQd = m$, whence

$$f = \frac{m}{Qd} = \frac{800}{10\,000 \times 0.4} = 0.2.$$

Problem 35. Referring to Fig. 90, an inclined ($\alpha = 30°$) wooden plank carries a load $G = 1.5$ kN enclosed in a wooden box. Determine the magnitude of the horizontal force **P** from the condition that the load is in equilibrium on the inclined plane. The coefficient of friction (wood on wood) f is taken equal to 0.4.

Solution. First determine P_{min}, which is the minimal value of the force **P** necessary to prevent the body from sliding down the plane. The force of friction will be directed opposite to that of possible motion, that is, upwards along the plane in our case.

Thus, the body is in equilibrium under the action of the forces **G**, **P**, **F**, and $\mathbf{R}_n$, which is the normal reaction of the plane.

Projecting all forces on the y- and x-axis that have the directions sndicated in Fig. 90, we get

$$\sum Y_k = R_n - G\cos\alpha - P_{\text{min}}\sin\alpha = 0,$$

whence

$$R_n = G\cos\alpha + P_{\text{min}}\sin\alpha,$$

$$\sum X_k = P_{\text{min}}\cos\alpha + F - G\sin\alpha = 0.$$

The force of friction $F = fR_n = f\,(G\cos\alpha + P_{\text{min}}\sin\alpha)$. Substituting its value into the preceding equation, we obtain

$$P_{\text{min}}\cos\alpha + fG\cos\alpha + fP_{\text{min}}\sin\alpha - G\sin\alpha = 0.$$

Solving this equation for P_{min} and substituting the data, we find

$$P_{\text{min}} = \frac{G(\sin\alpha - f\cos\alpha)}{\cos\alpha + f\sin\alpha} = \frac{1500\,(0.5 - 0.4\times 0.866)}{0.866 + 0.4\times 0.5} \approx 216\ \text{N}.$$

If the magnitude of the force **P** is less than P_{min}, then the body will begin to slide downwards. If the magnitude of the force **P** is increased, then the equilibrium of the body on the plane will remain until the maximum value P_{max} is reached, because the force **F** of friction will then be directed downwards along the plane in a direction opposite to that shown in Fig. 90 (the force of friction is always in the opposite direction to possible motion). The equations of equilibrium will remain the same, only the sign in front of the projection of the force **F** on the x-axis will be reversed, and we will have

$$P_{\text{max}} = \frac{G(\sin\alpha + f\cos\alpha)}{\cos\alpha - f\sin\alpha} \approx 1900\ \text{N}.$$

Combining both solutions thus obtained, we see that the load will remain in equilibrium on the inclined plane for all values of the magnitude of the force **P** that lie between the limits 216 N and 1900 N. When $P > 1900$ N, the load will move up the plane, and for $P < < 216$ N, it will move down. For any value of **P** greater than 216 N but less than 1900 N, this force will suffice to prevent the load from going downwards, but will be too small to make it move upwards.

It is necessary to note that, generally, the conditions of equilibrium of bodies in the presence of friction are expressed by inequalities, and so equilibrium problems have a set of solutions instead of one. A definite solution is obtained only when considering limiting cases of equilibrium in which the body is on the boundary line between relative rest and motion or when the body is in motion.

Problem 36. Referring to Fig. 91*a*, the rolls of a rolling mill rotate in opposite directions and pull in the sheet being rolled due to friction between the rolls and the sheet. What must the distance a between the rolls be in order to be able to roll sheets of thickness $b = 10$ mm? The radius of the rolls is $r = 30$ cm, and the coefficient of friction for incandescent iron and iron rolls $f = 0.1$.

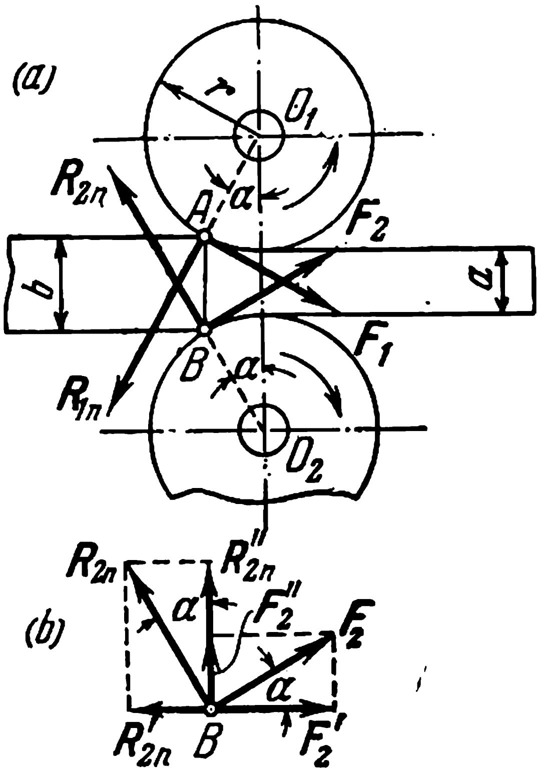


FIG. 91

Solution. The following forces are applied to the sheet: the normal reactions of the rolls, $\mathbf{R}_{1n}$ and $\mathbf{R}_{2n}$ directed along the radii O_1A and O_2B, and the friction forces $\mathbf{F}_1$ and $\mathbf{F}_2$ directed along tangents to the rolls at points A and B. By virtue of symmetry, it is obvious that $R_{1n} = R_{2n}$ and hence the friction forces $F_1 = fR_{1n}$ and $F_2 = fR_{2n}$ are equal in magnitude. From now on we put $R_{1n} = R_{2n} = R_n$ and $F_1 = F_2 = F = fR_n$. The normal reaction $\mathbf{R}_n$ of each roll is decomposed (Fig. 91*b*) into a vertical component $R''_n = R_n \cos\alpha$ and a horizontal component $R'_n = R_n \sin\alpha$. Similarly decomposed are the friction forces: into a horizontal component $F' = F\cos\alpha = fR_n \cos\alpha$, and into a vertical component $F'' = F\sin\alpha = fR_n \sin\alpha$. For the process of pulling in the sheet and subsequent rolling, it is obviously necessary that the sum of the horizontal components of the forces of friction be not less than the sum of the corresponding components of the normal reactions of the rolls, that is, it is necessary that we have the inequality $F' \geqslant R'_n$ or $fR_n \cos\alpha \geqslant R_n \sin\alpha$, whence $f \geqslant \tan\alpha$. Let us express $\tan\alpha$ in terms of the given dimensions and the thickness of the sheet. From Fig. 91*a* we have

$$O_1O_2 = 2r + a = b + 2r\cos\alpha,$$

whence

$$\cos\alpha = \frac{2r + a - b}{2r}.$$

Expressing $\cos\alpha$ in terms of $\tan\alpha$ and replacing that latter by f, we obtain

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \geqslant \frac{1}{\sqrt{1+f^2}}$$

or

$$\frac{1}{\sqrt{1+f^2}} \leqslant \cos\alpha.$$

Substituting this value into the preceding equation, we find

$$\frac{1}{\sqrt{1+f^2}} \leqslant \frac{2r+a-b}{2r},$$

whence we determine the desired distance between the rolls (the roll gap):

$$a \geqslant \frac{2r}{\sqrt{1+f^2}} + b - 2r.$$

Substituting the data, we obtain $a \geqslant \frac{2 \times 300}{\sqrt{1+0.1^2}} + 10 - 2 \times 300$ or $a \geqslant 7$ mm.

Sec. 38. The angle and cone of friction

Imagine a body resting on a rough surface (Fig. 92). If the surface were absolutely smooth, it would represent an ideal constraint whose action on the body would be reduced, as we know, to a single normal reaction, $\mathbf{R}_n$. But if the support surface is rough, then an additional frictional force appears, which lies in the tangential plane and is opposite to the direction in which we move or attempt to move the body. If we examine the critical instant (when the body lies at the brink between rest and motion), it will be found that the force of friction will have a maximum value, $F = fR_n$. Two reactions, the normal reaction $\mathbf{R}_n$ and the tangential (friction force) reaction $\mathbf{F}$ combine via the parallelogram rule and yield the total reaction $\mathbf{R}$ of the support surface, which now constitutes a certain angle φ with the normal to that surface.

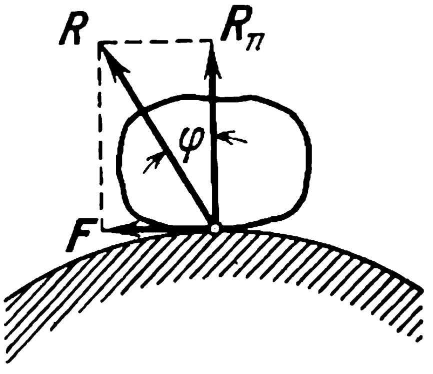


FIG. 92

The maximum angle φ *at which (due to friction) the reaction* **R** *of a rough surface deviates from the normal is called the angle of friction.*

From Fig. 92 we have

$$\tan\varphi = \frac{F}{R_n}.$$

But, as is evident from formula (33),

$$\frac{F}{R_n} = f.$$

Consequently,

$$\tan\varphi = f. \tag{34}$$

The tangent of the angle of friction is equal to the coefficient of sliding friction.

If we consider a body that can move along a rough support surface in any direction, then the lines of action of the possible reactions **R** of that surface form (Fig. 93) a conical surface.

The cone whose generators are inclined at the angle of friction φ *to the normal to the sliding surface at a given point is termed the cone of friction.*

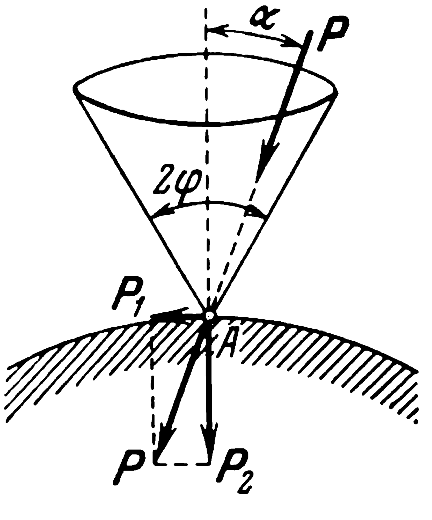


FIG. 93

If the coefficient of kinetic friction of a body is the same in different directions over a given surface, the total reaction **R** of the surface deviates from the normal in all directions through the same angle of friction φ and the cone of friction is a circular cone with angle at the vertex equal to 2φ. But if, as sometimes occurs (for instance, in friction on wood with and across the grain), the coefficient of kinetic friction of the body in different directions has different values, then the cone of friction will not be circular.

Referring again to Fig. 93, let the forces acting on the body (including the weight of the body) be reduced to a single resultant force **P** that passes through point A (the point of contact of the body with the surface) and forming with the normal to the surface at that point an angle α. Now translate that force along the line of its action to point A and decompose it into two components: $\mathbf{P}_1$ lying in the tangential plane, and $\mathbf{P}_2$ directed along the normal to the surface.

Then, by formulas (33) and (34), the maximal value of the static force of friction is

$$F = fP_2 = P_2 \tan \varphi,$$

where φ is the angle of friction and P_2 is the magnitude of the force of normal pressure of the body on the surface (which is clearly equal to the magnitude R_n of its normal reaction).

The magnitude of the force $\mathbf{P}_1$ that tends to make the body slide over the surface is $P_1 = P_2 \tan \alpha$.

For the body to remain on the surface in equilibrium, it is necessary that the condition $P_1 \leqslant F$ hold, or, if P_1 and F are substituted into this inequality, $P_2 \tan \alpha \leqslant P_2 \tan \varphi$, whence we obtain that the condition of equilibrium of the body on the surface is $\alpha \leqslant \varphi$. If the magnitude of the force $\mathbf{P}$ is increased and its direction is left unchanged, then there will be a proportional increase not only in the magnitude P_1 of the moving force, but also in the magnitude P_2 of the force of normal pressure. Now this inevitably implies a corresponding increase in the friction force, and the body will as before remain in equilibrium. The body comes into motion only when the magnitude of the force $\mathbf{P}_1$ becomes greater than the magnitude of the force $\mathbf{F}$, and to achieve this we have to change the direction of the force $\mathbf{P}$ so that the angle α becomes greater than the angle of friction φ, that is, so that the force $\mathbf{P}$ passes outside the cone of friction. Consequently, *if the resultant* $\mathbf{P}$ *of all forces applied to a body* (no matter what its magnitude) *passes inside the cone of friction, the body remains at rest; motion occurs only if the resultant passes outside the cone of friction.* It is this remarkable property of the region located inside the cone of friction that explains the name of the term.

Problem 37. A plane OA (Fig. 94) turns on a hinge O so that it may be set at any angle to the horizon. A body B weighing G is put on the plane. For what maximum angle α of inclination of the plane will the body remain in equilibrium?

Solution. The body remains in equilibrium on the plane under the action of the following forces: $\mathbf{G}$, the force of gravity, the normal reaction $\mathbf{R}_n$ of the plane, and the friction force $\mathbf{F}$. Projecting these forces on the chosen coordinate axes, x and y, we obtain the equations of equilibrium:

$$\sum X_k = F - G \sin \alpha = 0,$$
$$\sum Y_k = R_n - G \cos \alpha = 0.$$

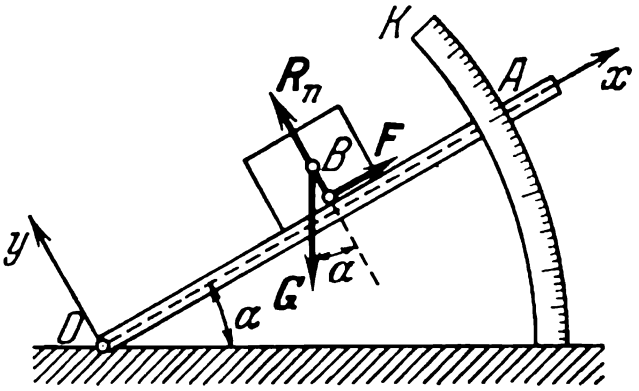


FIG. 94

From these equations we find

$$\tan\alpha = \frac{F}{R_n} = f = \tan\varphi$$

where φ is the angle of friction.

For equilibrium of the body it is thus necessary to observe the conditions $\alpha \leqslant \varphi$ or $\alpha \leqslant \arctan f$.

This relationship yields one of the methods of determining the coefficient of sliding friction at rest. Gradually increasing the angle of inclination of the plane, we note (on the scale K) the angle α at which the body begins to slide along the plane. The tangent of that angle gives us the desired coefficient of friction for the appropriate materials (the body and the plane).

A similar solution is provided for the problem of determining the so-called angle of repose of soil, that is, the maximum angle α of inclination of the soil to the horizon under which the particles of soil of the bank remain in equilibrium $\alpha = \varphi = \arctan f$, where f is the coefficient of friction among the particles of soil.

Sec. 39. Rolling friction

Rolling friction is a force that opposes the motion of any body rolling over the surface of another body.

This opposition (resistance) arises chiefly because both the rolling body and the body over which it rolls are not absolutely rigid and therefore undergo a certain deformation at the point of contact.

If (see Fig. 95) a cylindrical roller lying on a horizontal plane is under the sole action of the normal force **G**, then deformation of the roller and of the support plane is symmetric about the line of action of the force **G**. By reducing the reactions of the plane that are distributed over the small area of contact of the roller and plane to a resultant $\mathbf{R}_n$, we will always obtain the resultant equal in magnitude and opposite in direction to the force **G**.

If (see Fig. 96) the roller is acted upon at a certain height *h* by yet another force, a horizontal force **P**, then the deformation of the roller and the support plane will no longer be

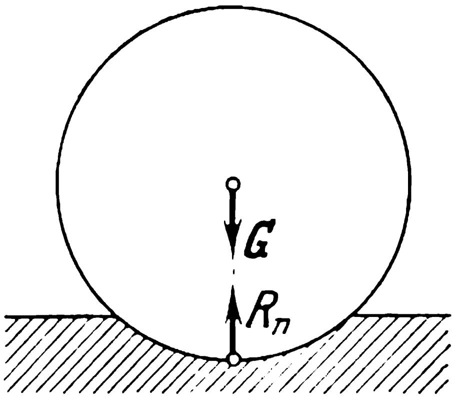


FIG. 95

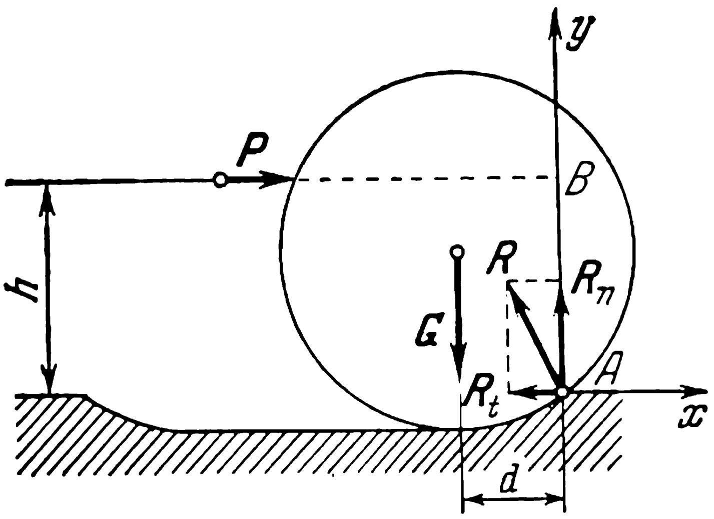


FIG. 96

symmetric about the line of action of the force **G**. The resultant **R** of the reactions of the plane distributed over the area of contact with the roller will shift towards possible motion of the roller and will be directed along the normal to the surface of contact at some point *A*.

Let us resolve this unknown (in magnitude) reaction **R** into a horizontal component $\mathbf{R}_t$ and a vertical component $\mathbf{R}_n$ and apply the conditions of equilibrium to the plane system of forces **P**, **G**, $\mathbf{R}_n$ and $\mathbf{R}_t$ impressed on the roller:

$$\sum X_k = P - R_t = 0,$$

whence

$$R_t = P,$$

$$\sum Y_k = -G + R_n = 0,$$

from which we get

$$R_n = G,$$

$$\sum m_A (\mathbf{P}_k) = Gd - P \times AB = 0.$$

Assuming $AB = h$ due to the smallness of the deformation and replacing G by the numerically equal value R_n, we get

$$Ph = R_n d.$$

The quantity $R_n d$ *is called the moment of rolling friction. Experiment shows that the moment of rolling friction increases with increase in the rotational moment* Ph *but cannot exceed a certain value which is very definite for the given pair of surfaces in contact and for the given force of normal pressure of the roller on the surface:*

$$R_n d \leqslant kR_n,$$

where k is a proportionality factor called the *coefficient of rolling friction.*

We thus obtain the following *condition for equilibrium of a roller on a horizontal plane:*

$$Ph \leqslant kR_n,$$

whence

$$P \leqslant \frac{k}{h} R_n, \qquad (35)$$

where P is the magnitude of the horizontal force applied to the roller, k is the coefficient of rolling friction for the given pair of surfaces (roller and support plane), h is the height of application of the horizontal force, R_n is the magnitude of the normal reaction of the plane or the magnitude (equal to it) of the force of normal pressure of the roller on the plane.

For the case of the limiting equilibrium of the roller,

$$P = \frac{k}{h} R_n. \qquad (36)$$

Assuming, as is customary, that the coefficient of friction of the roller remains unchanged in the rolling process, use can also be made of formula (36) to determine the force necessary for a uniform rolling of the roller.

The coefficient of rolling friction depends on the elastic properties of the materials of the rubbing bodies and on the state of their surfaces. It is a constant quantity for a given pair of rubbing bodies.

The following table gives approximate values of the coefficients of rolling friction (for a roller on a surface).

Materials of rubbing bodies	Coefficient of friction k (cm)
Soft steel on soft steel	0.005
Hardened steel on hardened steel	0.001
Iron on iron	0.005
Wood on steel	0.03-0.04
Wood on wood	0.05-0.08

Unlike the coefficient of sliding friction, which is a dimensionless number, the coefficient of rolling friction, which constitutes the maximum value of displacement d (Fig. 96) of the normal reaction $\mathbf{R}_n$ of the support plane, is measured in units of length; the values of k and h in formulas (35) and (36) must be in the same units.

For the most common practical case of application of force **P** to the axis of a cylindrical roller, assuming $h=r$, we have the following condition of uniform rolling:

$$P=\frac{k}{r}R_n,$$

where r is the radius of the cylinder.

For the case where the force **P** is applied at a height equal to the diameter of the roller (for example, when heavy loads are being carried on rollers), assuming $h=d$, we have

$$P=\frac{k}{d}R_n,$$

where d is the diameter of the cylinder.

In most cases, rolling friction is many times less than sliding friction, and that is why sliding is replaced wherever possible by rolling in practical situations. For example, whenever a heavy object has to be moved, it is not dragged along the surface (sliding) but rather it is rolled on rolls put under it.

The same principle of replacing sliding friction with rolling friction is used in such common components as roller bearings and ball bearings. The advantage of these bearings over sliding bearings, lies in the fact that, besides the smaller losses due to friction, their resistance to starting is almost equal to the resistance in the case of established motion (this is because rolling friction is almost independent of velocity).

It should be noted that, generally, a cylinder may not only roll over the support plane but also slide along it.

1. If $P \geqslant \frac{k}{h} R_n$, then $P < fR_n$ (where f is the coefficient of sliding friction), the cylinder will only roll.

2. If $P \geqslant fR_n$, but $P < \frac{k}{h} R_n$ (which is a very rare occurrence—only for a very small value of h), the cylinder will only slide.

3. If $P \geqslant \frac{k}{h} R_n$ and $P \geqslant fR_n$, then both rolling and sliding of the cylinder are possible.

Problem 38. Figure 97 shows a ball step bearing onto which is conveyed a force of pressure $Q = 100$ kN; the ball diameter $d = 40$ mm. Determine the magnitude of the horizontal force **P** necessary to overcome friction in the step bearing assuming it to be applied at a distance from the axis of the shaft equal to the mean radius of the ring. The coefficient of rolling friction of the balls made of hardened steel on steel is $k = 0.001$ cm.

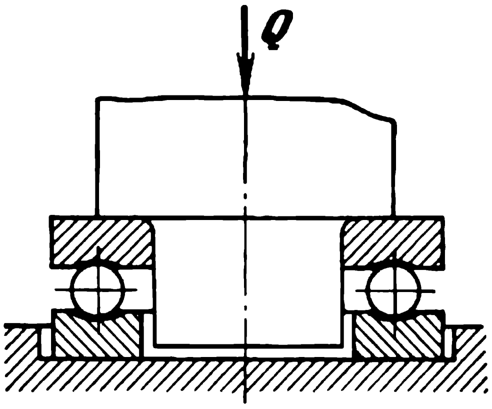


FIG. 97

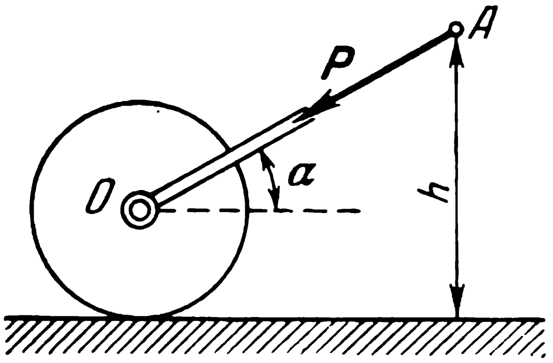


FIG. 98

Solution. In this case we have rolling friction on the one hand between the balls and the surface of the fixed ring of the step bearing, on the other hand between the balls and the surface of the moving ring (fixed to the shaft) of the step bearing. The force **P** is transferred to the balls at a height equal to their diameters. Hence, the desired force is

$$P = 2\frac{k}{h} R_n = 2\frac{k}{d} Q = 2\frac{0.01}{40} \times 100\,000 = 50 \text{ N}.$$

Problem 39. Referring to Fig. 98, a cylindrical roller of diameter $d = 60$ cm and weight $G = 3$ kN is set into motion along a horizontal plane by a man pressing on lever AO with a constant force $\mathbf{P}$ in the direction AO; $AO = 1.5$ m in length, and point A is $h = 1.05$ m above the horizon. Determine the magnitude of the force $\mathbf{P}$ needed for uniform rolling of the roller if the coefficient of rolling friction $k = 0.5$ cm.

Solution. Let us first find the angle made by the force $\mathbf{P}$ with the horizontal plane:

$$\sin\alpha = \frac{h-r}{AO} = \frac{1.05-0.3}{1.5} = 0.5, \qquad \alpha = 30°.$$

Decompose the force $\mathbf{P}$ into a horizontal component $\mathbf{P}_1$ that brings the roller into motion, and a vertical component $\mathbf{P}_2$ that causes an increase in the normal pressure of the roller on the horizontal plane.

The horizontal force P_1 applied to the axis of the roller is found from formula (36):

$$P_1 = P\cos\alpha = \frac{k}{r}R_{\text{n}}.$$

The normal reaction of the horizontal plane is in this case

$$R_{\text{n}} = G + P_2 = G + P\sin\alpha,$$

whence

$$P\cos\alpha = \frac{k}{r}(G + P\sin\alpha)$$

and

$$P = \frac{kG}{r\cos\alpha - k\sin\alpha} = \frac{0.5\times 3000}{30\times 0.866 - 0.5\times 0.5} = 57 \text{ N}.$$

CHAPTER VII

A SPATIAL SYSTEM OF FORCES

Sec. 40. A spatial system of converging forces

In many practical cases, all forces acting on a structure or mechanism are reduced to a system of forces lying in one plane. Examples of this appeared in all the problems we have so far worked on. It often happens that a spatial system of forces can be broken down into several plane systems.

However, this is by far not always the case. For this reason, in statics we have to study the reduction and equilibrium of systems of forces not lying in the same plane, so-called *spatial systems of forces.*

A system of several forces whose lines of action do not lie in the same plane but intersect at one point is termed a *spatial system of converging forces.* As in the case of a plane system of converging forces, this system too can be reduced to a system of forces applied at one point. Suppose (see Fig. 99) several forces not lying in one plane are applied at point *O*; for example, the four forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Although all the forces do not lie in the same plane, every two of the forces pairwise do definitely lie in some one plane for the simple reason that through any two intersecting straight lines it is always possible to pass a plane, and only one plane at that. Thus, to combine any two forces of such a system, for instance in order to combine $\mathbf{P}_1$ and $\mathbf{P}_2$, we can apply the rule for the composition of converging forces in a plane and find their resultant $\mathbf{R}_1$ applied to the same point *O*. Drawing a plane through that resultant and some third force $\mathbf{P}_3$, we can, using the same triangle rule, find the resultant $\mathbf{R}_2$ of three forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and so forth.

As is evident from Fig 99, the *resultant of a spatial system of converging forces is depicted in magnitude and direction by the closing side of the polygon (OABCD) constructed on the component forces, thus constituting their geometric sum*:

$$\mathbf{R}_P = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4$$

or

$$\mathbf{R}_P = \sum \mathbf{P}_k. \tag{37}$$

It is necessary to bear in mind that the force polygon resulting from the composition of a spatial system of forces will not be plane. A nonplane polygon can be constructed by, say, wire.

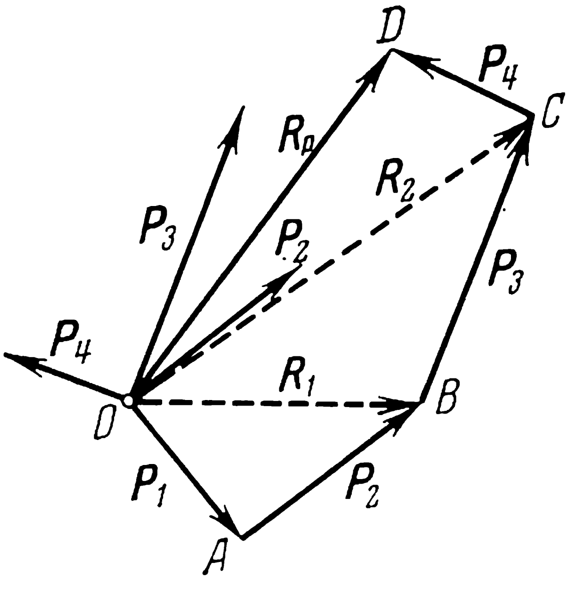


FIG. 99

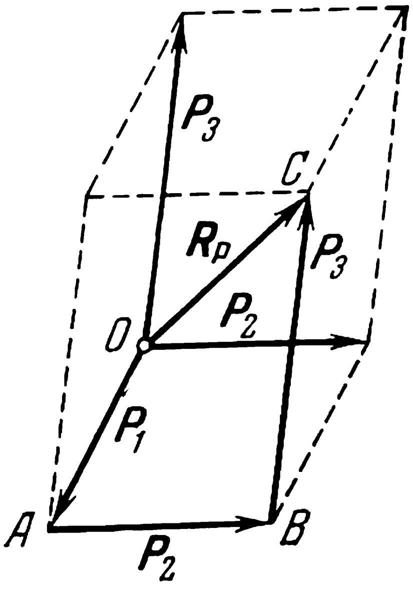


FIG. 100

In a particular case, the *resultant of a spatial system of three converging forces is depicted in magnitude and direction by the diagonal of a parallelepiped constructed on the forces. (The rule of a parallelepiped of forces.)*

Indeed, referring to Fig. 100, for three forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$, the diagonal *OC* of the corresponding parallelepiped is the closing side *OC* of the spatial polygon *OABC* of those forces.

By proceeding from the parallelepiped rule, it is easy to solve the converse problem: that of decomposing the force into three specified directions not lying in one plane. To do this, it obviously suffices to construct a parallelepiped whose edges have the specified directions and whose diagonal is the specified force.

If the three specified directions are the directions of the coordinate axes (Fig. 101), then the components $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ of the force $\mathbf{P} = \overrightarrow{OA}$ are correspondingly equal in magnitude to the absolute values of the projections of the given force on the axes of the spatial system of coordinates:*

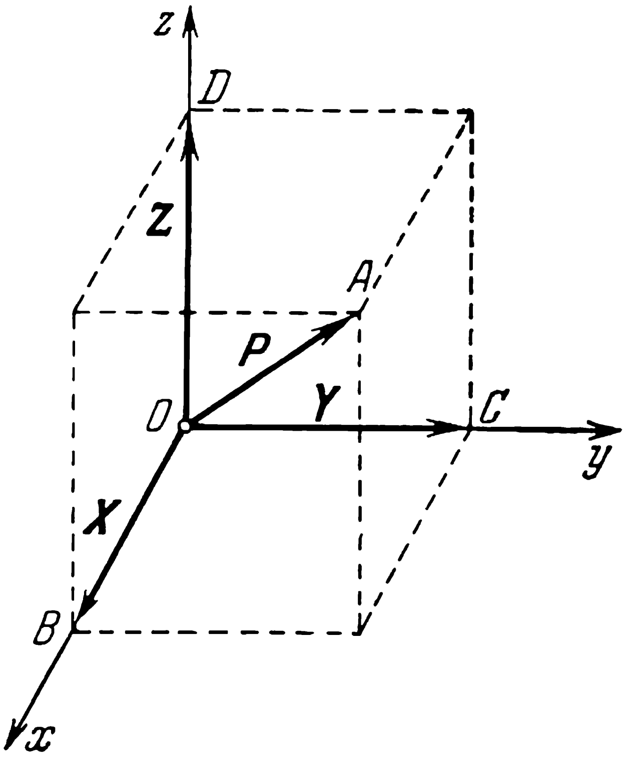


FIG. 101

$$OB = |\mathbf{X}|, \quad OC = |\mathbf{Y}|$$

and

$$OD = |\mathbf{Z}|$$

whence

$$P = \sqrt{X^2 + Y^2 + Z^2}. \quad (38)$$

The magnitude of the force is equal to the square root of the sum of the squares of its projections on any three mutually perpendicular axes.

Proof- was given earlier (Sec. 12) that the projection of a geometric sum of vectors on any axis is equal to the alge-

---

* We will make use of the right-hand system of coordinates. The axes in this system (Fig. 102) can be determined by the fingers of the right hand: the thumb and first two fingers lie along the three mutually perpendicular directions, the thumb indicating the direction of the x-axis, the forefinger that of the y-axis, and the middle finger, the z-axis. In the right-hand system, passing from the x-axis to the y-axis to the z-axis is in the counterclockwise direction. This sense of rotation is taken as positive when using the right-hand system of coordinates,

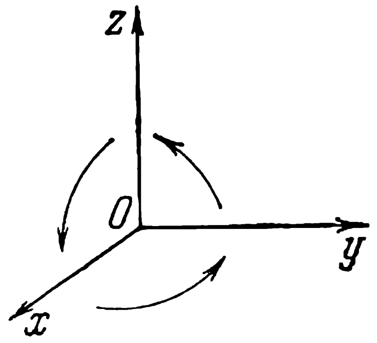


FIG. 102

braic sum of the projections of the component vectors on the same axis. Since this proposition holds true for any (plane or spatial) vectorial polygon and the resultant of a system of converging forces is equal to the geometric sum of the vectors of the component forces, the projection of the resultant of the system of converging forces on any axis is equal to the algebraic sum of the projections of the component forces on that axis:

$$R_{Px} = \sum X_k, \quad R_{Py} = \sum Y_k, \quad R_{Pz} = \sum Z_k.$$

Substituting these values into formula (38), we get

$$R_P = \sqrt{(\sum X_k)^2 + (\sum Y_k)^2 + (\sum Z_k)^2} \tag{39}$$

which is an expression for the magnitude of the resultant of a spatial system of converging forces.

Since the projection of a force on some axis is equal to the magnitude of that force multiplied by the cosine of the angle between the direction of the force and the positive direction of the axis of projections, it follows that

$$R_{Px} = R_P \cos(\widehat{\mathbf{R}_P, x}), \quad R_{Py} = R_P \cos(\widehat{\mathbf{R}_P, y}),$$

$$R_{Pz} = R_P \cos(\widehat{\mathbf{R}_P, z}).$$

From this we obtain formulas that determine the direction of the resultant:

$$\left.\begin{aligned} \cos(\widehat{\mathbf{R}_P, x}) &= \frac{R_{Px}}{R_P} = \frac{\sum X_k}{R_P}, \\ \cos(\widehat{\mathbf{R}_P, y}) &= \frac{R_{Py}}{R_P} = \frac{\sum Y_k}{R_P}, \\ \cos(\widehat{\mathbf{R}_P, z}) &= \frac{R_{Pz}}{R_P} = \frac{\sum Z_k}{R_P}. \end{aligned}\right\} \tag{40}$$

If the system of converging forces is in equilibrium, then its resultant $\mathbf{R}_P$ must clearly be zero.

Since the resultant is the closing side of a force polygon, it follows that *for the equilibrium of a spatial system of converging forces it is necessary and sufficient that the force polygon constructed on the given forces be closed.* (*The condition of* *equilibrium in geometric form.)* From formula (39), it is clear that $R_P = 0$ when the following equations occur:

$$\left.\begin{aligned}\sum X_k &= 0,\\ \sum Y_k &= 0,\\ \sum Z_k &= 0.\end{aligned}\right\} \qquad (41)$$

For the equilibrium of a spatial system of converging forces, it is necessary and sufficient that the sums of the projections of all forces on each of any three mutually perpendicular axes be zero.

In addition to what was said in Sec. 11 on the projection of a vector on an axis, we may note that in order to find the projection of a vector on an axis not lying in the same plane as the vector it is sometimes more convenient to first project the vector on the plane in which the given axis lies, and then project on the given axis the projection (just found) of the vector on the plane (the method of double projection). Thus, for example, the projection of vector **Q** on the x-axis (Fig. 103) is

$$Q_x = aD = ab\cos\beta = \\ = Q_{\text{Pr}}\cos\beta = Q\cos\alpha\cos\beta. \quad (42)$$

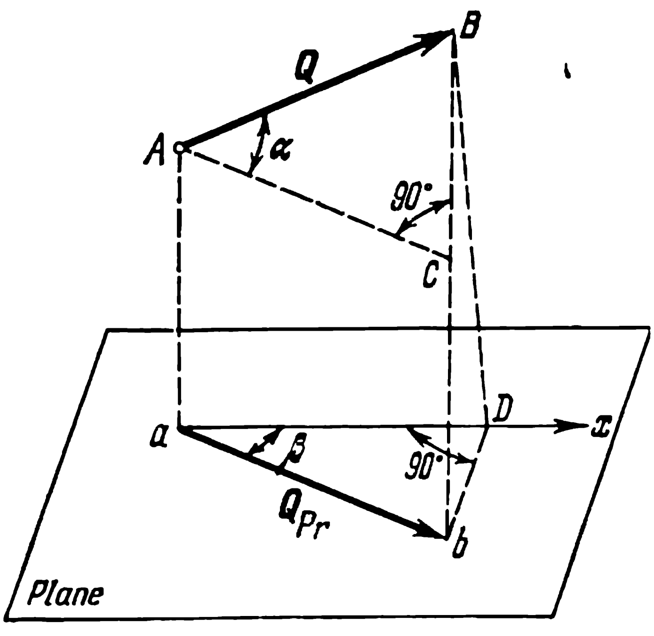


FIG. 103

It has already been established that the projection of a vector on an axis is a scalar algebraic quantity. Now the projection of a vector on a plane is a vector quantity and, hence, is characterized not only by its numerical value but also by its direction in the plane of projection. In magnitude, the projection of a vector on a plane (Fig. 103) is

$$Q_{\text{Pr}} = ab = AC = AB\cos\alpha = Q\cos\alpha,$$

where α is the angle between the direction of the vector **Q** and the direction of its projection $\mathbf{Q}_{\text{Pr}}$ on the plane.

Problem 40. Referring to Fig. 104, a shaft is being machined. With the aid of a dynamometer it is found that in the direction of longitudinal feed the cutting tool experiences a resistance (axial force) $P_y = 1$ kN, in the direction of cross feed (radial force) $P_x = 2.2$ kN, and in the vertical direction, a resistance $P_z = 5$ kN. Determine the total force of pressure on the cutting tool.

Solution. Since the component forces are directed along three mutually perpendicular straight lines, the total force of pressure on the cutting tool is depicted as the diagonal of a right parallelepiped constructed on the components of force. From this we find the magnitude of the total pressure:

$$R_P = \sqrt{P_x^2 + P_y^2 + P_z^2} = \sqrt{2.2^2 + 1^2 + 5^2} \approx 5.55 \text{ kN}.$$

The direction of this force is found from the formulas

$$\cos(\widehat{\mathbf{R}_P, x}) = \cos\alpha = \frac{P_x}{R_P} = \frac{220}{555},$$

$$\cos(\widehat{\mathbf{R}_P, y}) = \cos\beta = \frac{P_y}{R_P} = \frac{100}{555},$$

$$\cos(\widehat{\mathbf{R}_P, z}) = \cos\gamma = \frac{P_z}{R_P} = \frac{500}{555},$$

whence we find $\alpha = 66°40'$, $\beta = 79°40'$ and $\gamma = 25°40'$.

Problem 41. Referring to Fig. 105, at point A (where three bars AB, AC and AD are fixed in position) is suspended a load weighing

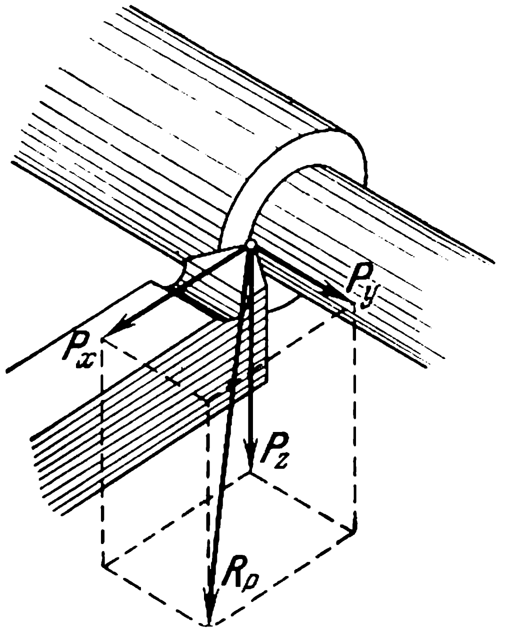


FIG. 104

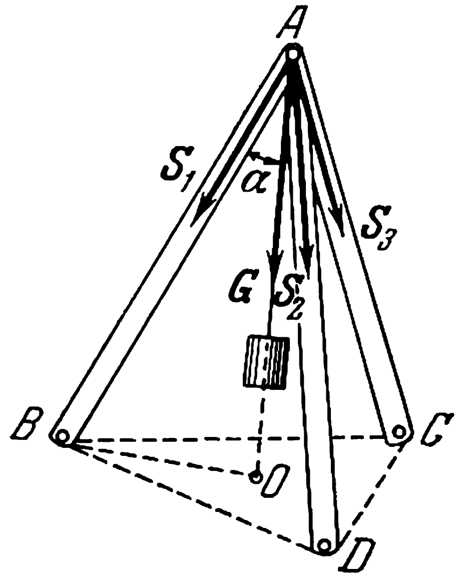


FIG. 105

$G = 1$ kN. Find the forces compressing each of the bars if the distance of A from the ground, AO, is 8 m, and each bar is 10 m long. The points B, C, and D lie at the vertices of an equilateral triangle.

Solution. To determine the desired forces, it suffices to decompose the force G into three components directed along AB, AC and

AD; due to symmetry, these components are equal. Denote the magnitude of each compressing force by S, the angle of each of the bars with the vertical by α.

Projecting the force G and its three components S_1, S_2, S_3 (all with the same magnitude S) on the vertical, we obtain

$$G = 3S \cos \alpha$$

(by the theorem on the projection of the resultant), whence

$$S = \frac{G}{3 \cos \alpha}.$$

We now determine the angle α. From the right triangle OAB we have

$$\cos \alpha = \frac{OA}{AB} = \frac{8}{10} = 0.8.$$

Consequently,

$$S = \frac{1000}{3 \times 0.8} = 417 \text{ N}.$$

Problem 42. Referring to Fig. 106, a rod CD is hinged at one end to a vertical wall. The other end is held by two horizontal chains, AD and BD, of the same length. The chains are attached to the same wall. A load of weight $G = 1$ kN is suspended at the end D. Find the stress in the rod and the tension of the chains if $\alpha = 60°$ and $\beta = 45°$.

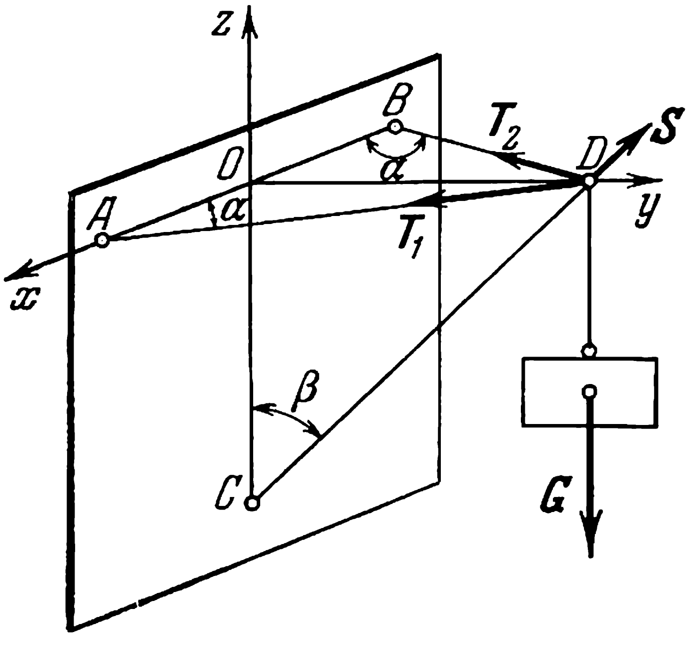


FIG. 106

Solution. The force of gravity G of the load acts on point D. This point is not free; the constraints are the rod CD and the chains AD and BD. Let us get rid of the constraints by replacing them by the reactions S (of the rod) and T_1 and T_2 (of the chains AD and BD).

The lines of action of these reactions are known: they coincide with the straight lines CD, AD and BD. It is clear that the rod CD

is compressed* and therefore its reaction **S** is directed away from C towards point D. The chains AD and BD are under tension and therefore their reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ are directed from D to A and from D to B. Thus, four forces converge at point D: **G**, **S**, $\mathbf{T}_1$ and $\mathbf{T}_2$.

These forces are in equilibrium and must satisfy the equations of equilibrium (41). Draw the coordinate axes as shown in Fig. 106, bringing to coincidence plane yOz with the plane in which the forces **G** and **S** are acting. Then the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ will lie in the coordinate plane xOy and the projections of the forces on the axes will be found in the simplest possible manner.

Force	Projection of force on the axes		
	x	y	z
G	0	0	$-G$
S	0	$S\cos(90^\circ-\beta)=S\sin\beta$	$S\cos\beta$
$\mathbf{T}_1$	$T_1\cos\alpha$	$-T_1\cos(90^\circ-\alpha)=-T_1\sin\alpha$	0
$\mathbf{T}_2$	$-T_2\cos\alpha$	$-T_2\cos(90^\circ-\alpha)=-T_2\sin\alpha$	0

Let us project all the given forces on the coordinate axes (see table) and set up the equilibrium equations:

$$\sum X_k = T_1\cos\alpha - T_2\cos\alpha = 0,$$
$$\sum Y_k = S\sin\beta - T_1\sin\alpha - T_2\sin\alpha = 0,$$
$$\sum Z_k = -G + S\cos\beta = 0.$$

Substituting the numerical values and solving the system of equations, we find $S = 1.41$ kN, $T_1 = T_2 = 578$ N.

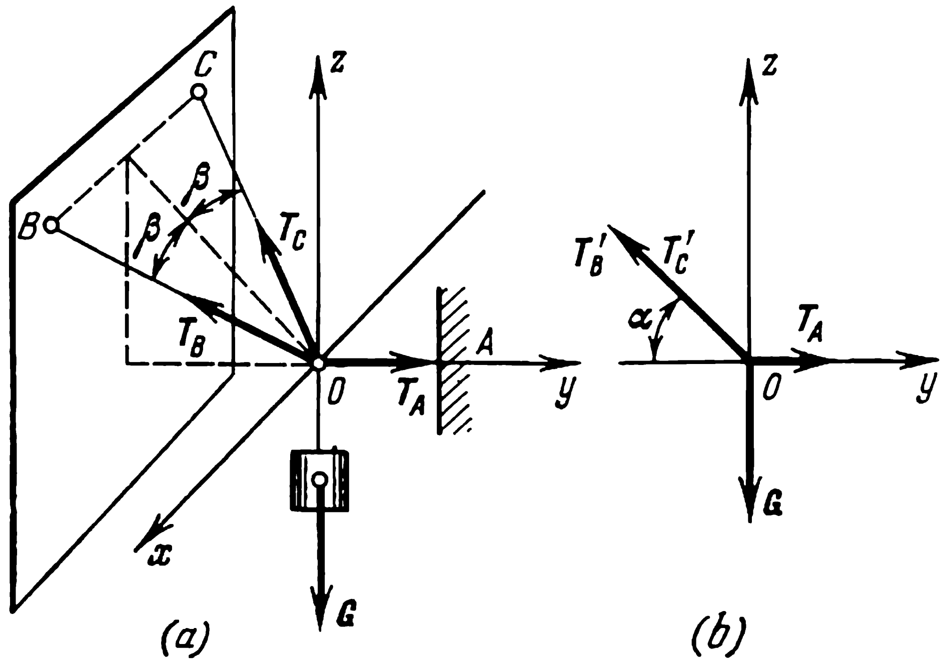
This problem, like other problems involving the equilibrium of spatial systems of converging forces, may be reduced to the problem of the equilibrium of a plane system of converging forces. From the solution it is evident that the reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ are equal in magnitude. This could have been foreseen earlier due to the symmetry in the location of the chains AD and BD. The resultant **T** of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ will clearly be directed along the y-axis from D to O, and the solution of the problem on the equilibrium of a spatial system of forces **G**, **S**, $\mathbf{T}_1$ and $\mathbf{T}_2$ could have been reduced to the solution of a problem on the equilibrium of the system of forces **G**, **S**, **T**, which lie in the single plane yOz. After having found the resultant **T**, it would have been easy to determine the reactions of the chains by a simple decomposition of the force **T** along the directions DA and DB.

---

* If it is not immediately apparent which of the rods is under compression and which is under tension, take it, preliminarily, that all rods are under tension. A negative value of the reaction of one or another rod obtained in the solution of the problem will indicate that the actual direction of that reaction is opposite the accepted direction, that is, the rod is under compression.

Problem 43. Referring to Fig. 107*a*, a load $G = 100$ N is held in equilibrium by three ropes: a horizontal rope AO and two ropes BO and CO, the plane of the latter two ropes being inclined to the horizontal plane at an angle of $\alpha = 45°$ and perpendicular to the vertical plane passing through the rope AO. The ropes BO and CO are symmetric about this vertical plane and form angles $\beta = 30°$ with it. Determine the tension of the ropes.

Solution. A spatial system of forces converges at point O: the force of gravity **G** of the load and the reactions $\mathbf{T}_A$, $\mathbf{T}_B$ and $\mathbf{T}_C$ of the



(a) (b)

FIG. 107

ropes OA, OB and OC. We choose the coordinate axes as shown in Fig. 107*a*, bringing plane yOz to coincidence with the vertical plane of symmetry.

The projections of the forces **G** and $\mathbf{T}_A$ on the coordinate axes are found easily. Now, when projecting the forces $\mathbf{T}_B$ and $\mathbf{T}_C$ on the axes y and z, we take advantage of the earlier mentioned (page 156) device of double projection. We first find the projections $\mathbf{T}_B'$ and $\mathbf{T}_C'$ of the forces $\mathbf{T}_B$ and $\mathbf{T}_C$ on the plane yOz, and then we project these projections on the y- and z-axis (Fig. 107*b*).

It is obvious that the magnitudes of the vectors of projections of the forces $\mathbf{T}_B$ and $\mathbf{T}_C$ on the yOz-plane will be equal to

$$T_B' = T_B \cos\beta \quad \text{and} \quad T_C' = T_C \cos\beta.$$

Let us now find the values of the projections of all forces on the coordinate axes.

Force	Projection of force on the axes		
	x	y	z
G	0	0	$-G$
$\mathbf{T}_A$	0	T_A	0
$\mathbf{T}_B$	$T_B \cos(90^\circ - \beta) =$ $= T_B \sin \beta$	$-T'_B \cos \alpha =$ $= -T_B \cos \beta \cos \alpha$	$T'_B \cos (90^\circ - \alpha) =$ $= T_B \cos \beta \sin \alpha$
$\mathbf{T}_C$	$-T_C \cos(90^\circ - \beta) =$ $= -T_C \sin \beta$	$-T'_C \cos \alpha =$ $= -T_C \cos \beta \cos \alpha$	$T'_C \cos (90^\circ - \alpha) =$ $= T_C \cos \beta \sin \alpha$

Setting up the equilibrium equations

$$\sum X_k = T_B \sin \beta - T_C \sin \beta = 0,$$

we get $T_B = T_C$. This might have been guessed directly from symmetry reasoning. The other two equations are of the form

$$\sum Y_k = T_A - T_B \cos \beta \cos \alpha - T_C \cos \beta \cos \alpha = 0,$$

$$\sum Z_k = -G + T_B \cos \beta \sin \alpha + T_C \cos \beta \sin \alpha = 0.$$

Substituting the numerical data and solving the system of equations, we obtain $T_B = T_C = 820$ N, $T_A = 100$ N.

Sec. 41. The moment of a force about an axis

The moment of a force about an axis is a quantity that characterizes the rotational effect of the given force about that axis.

We begin with a concrete example. Suppose a force **P** applied to a body that can rotate about a fixed axis, say, to a door that turns in its hinges about an axis z (Fig. 108), does not lie in a plane perpendicular to that axis.

It is clear that the force component **P**$''$, which is parallel to the axis of rotation of the body, cannot impart to the body a rotational motion; this force tends only to shift the body along the z-axis. The rotational effect is caused by the component **P**$'$, which is the projection of the given force **P** on the plane perpendicular to the axis of rotation of the body and lies in that plane. Consequently, the measure of the rotational effect of the action of force **P** about the z-axis is the moment of the projection of this force on a plane perpendicular to the z-axis about the point of intersection O

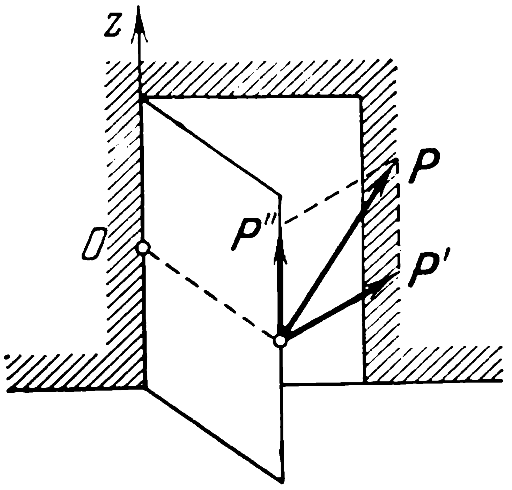


FIG. 108

of the axis with the indicated plane. This moment is called the moment of the force **P** about the axis.

Thus, *the moment of a force about an axis is equal to the moment of the projection of the force on a plane perpendicular to the given axis about the point of intersection of the axis and the plane.* The concept of the moment of a force about an axis is one of the most important concepts of spatial statics and should be firmly fixed in the student's mind. In order to find the moment of some force **P** about an axis, say the z-axis (Fig. 109), project the force on the plane that is perpendicular to the given z-axis and then take the moment of the projection about the point of intersection of the axis and the plane.

Denoting the moment of the force **P** about the z-axis by $m_z(\mathbf{P})$, the magnitude of the projection of the force **P** on the plane perpendicular to the z-axis by P_{Pr}, and the arm of this projection about the point of intersection of the axis and the plane by h, we have

$$m_z(\mathbf{P}) = m_O(\mathbf{P}_{\mathrm{Pr}}) = \pm P_{\mathrm{Pr}} h. \qquad (43)$$

The sign in this formula is determined by the following rule: if to an observer looking at the plane from the positive side of the z-axis (indicated by an arrow) the projection of the force **P** on the plane appears to be rotating about the z-axis counterclockwise, then the moment is taken to be positive,[*] otherwise it is negative.

The moment of a force about an axis is fully determined by its numerical value and by the sign and, hence, is a scalar algebraic quantity.

Note that

(1) *the moment of a force about a given axis remains unchanged if the force is translated along its line of action*, for in that case neither the projection of the force on the given plane nor its arm undergoes change;

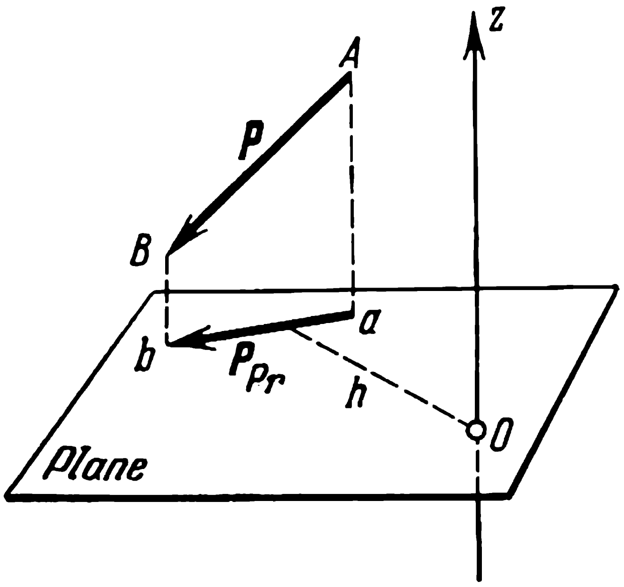


FIG. 109

(2) *the moment of a force about an axis is equal to zero if the line of action of the force and the axis lie in the same plane.*

Two cases are then possible.

(a) The force is parallel to the axis. In this case, the projection of the force on the plane perpendicular to the axis is zero.

(b) The liine of action of the force intersects the axis. Then the projection of the force on the plane passes through the point of intersection of the axis and the plane and its arm about that point is equal to zero.

Problem 44. Referring to Fig. 110, the downward branch of the belt acting on the circumference of a pulley of diameter $d = 400$ mm

---

[*] For a force **P** and the z-axis shown in Fig. 109,

$$m_z(\mathbf{P}) = +\mathbf{P}_{Pr}h.$$

with force $P = 400$ N deviates from the mean plane of the pulley MN through an angle $\alpha = 20°$. Determine the moment of the force about the axis OO_1 of the shaft.

Solution. Project the force **P** on the plane MN perpendicular to the axis of the shaft. The magnitude of the projection $P_{\mathrm{Pr}} = P \cos \alpha$.

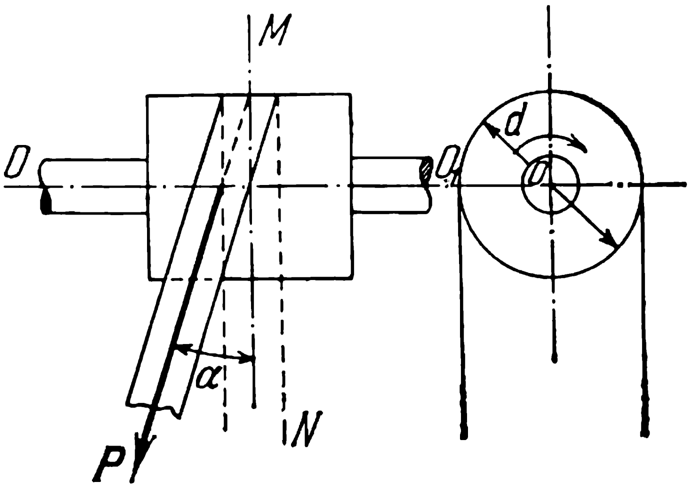


FIG. 110

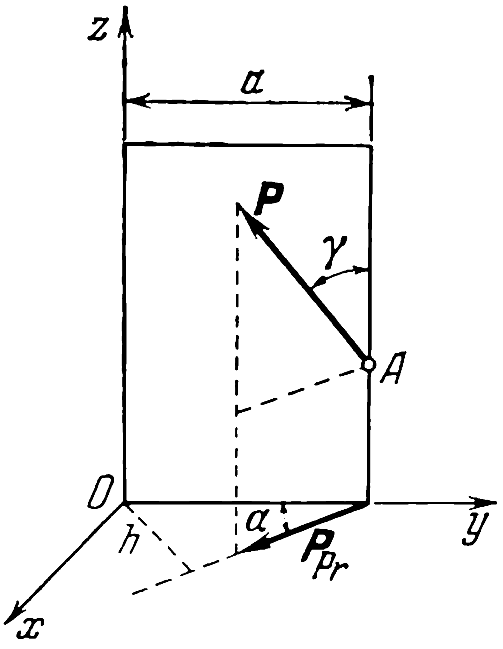


FIG. 111

Its distance to the point of intersection of the axis and the plane MN (that is, to the centre of the pulley) is equal to $d/2$. Thus the numerical value of the moment is

$$m_{OO_1}(\mathbf{P}) = P_{\mathrm{Pr}} \frac{d}{2} = P \frac{d}{2} \cos \alpha = 400 \times \frac{0.4}{2} \times \cos 20° \approx 75.2 \ \mathrm{N \cdot m}.$$

If the mean line of the portion of the belt under consideration coincided with the mean plane of the pulley, then the moment of that force about the axis of the shaft would be equal to

$$P \frac{d}{2} = 400 \times \frac{0.4}{2} = 80 \ \mathrm{N \cdot m}.$$

Problem 45. Referring to Fig. 111, a force $P = 20$ N is applied to a door that turns about the vertical axis Oz. The force is applied at the point A at an angle $\gamma = 60°$ to the vertical. The vertical plane in which the force lies forms an angle of $\alpha = 45°$ with the plane of the door. Determine the moment of the force **P** about the Oz-axis if the door is $a = 0.5$ m wide.

Solution. Draw a plane Oxy perpendicular to the axis Oz and project the force **P** on that plane; the magnitude of this projection, $P_{\mathrm{Pr}} = P \sin \gamma$. Drop a perpendicular from point O, the point of intersection of the axis with the plane, on the line of projection; the length of this perpendicular $h = a \sin \alpha$. Thus,

$$m_z(\mathbf{P}) = -P_{\mathrm{pr}}h = -aP \sin\alpha \sin\gamma$$

$$= -0.5 \times 20 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = -6 \text{ N}\cdot\text{m}.$$

Sec. 42. The conditions of equilibrium of a system of forces located at random in space

The method for reducing forces to a single centre that was considered in Sec. 26 for a plane system of forces is fully applicable also to a system of forces located at random in space.

According to the Poinsot theorem (Sec. 22), any force may be translated parallel to itself to any point of the body if a couple is adjoined in the process. By translating each of the forces of a spatial system to some one arbitrary point O (the centre of reduction), we obtain a spatial system of forces converging to one point and also a spatial system of couples, that is, a system of couples located in different planes.

The forces applied to a single point O may be combined by the rule of a force polygon and replaced by a single equivalent force $\mathbf{R}_{\mathrm{pr}}$ applied to the same point O:

$$\mathbf{R}_{\mathrm{Pr}} = \sum \mathbf{P}_k.$$

As in a plane system of forces, the *vector* $\mathbf{P}_{\mathrm{pr}}$ *is equal to the geometric sum of all given forces of the spatial system and is termed the principal vector of the system.*

The magnitude of this force may be computed via formula (39) that was earlier established for the magnitude of the resultant of a spatial system of converging forces:

$$R_{\mathrm{Pr}} = \sqrt{(\sum X_k)^2 + (\sum Y_k)^2 + (\sum Z_k)^2}.$$

The system of couples located in distinct planes that is obtained when reducing a spatial system of forces to a single centre may also be replaced by a single resultant couple, the moment $\mathbf{M}_O$ of which is called the *principal moment of* *the given spatial system of forces relative to the chosen centre of reduction O.*

There are ways of combining couples located arbitrarily in space. Through the use of special rules, one can find not only the magnitude of the resultant couple but also the plane in which the given couple will be located and the direction of rotation of the couple in that plane. But in order to set up those rules, one would have to introduce a number of concepts that go beyond the limits of the course of theoretical mechanics for technical schools.

It is demonstrated in deeper courses of mechanics that the magnitude M_O of the moment of the resultant couple (the magnitude of the principal moment of a spatial system of forces about a chosen centre O of reduction) is

$$M_O = \sqrt{[\sum m_x(\mathbf{P}_k)]^2 + [\sum m_y(\mathbf{P}_k)]^2 + [\sum m_z(\mathbf{P}_k)]^2},$$

where $\sum m_x(\mathbf{P}_k)$, $\sum m_y(\mathbf{P}_k)$ and $\sum m_z(\mathbf{P}_k)$ are the algebraic sums of the moments of all forces of the system about any three mutually perpendicular coordinate axes (with the origin of coordinates at the centre of reduction).

Since a couple cannot under any circumstances be balanced by a single force, it is necessary and sufficient, for the equilibrium of an arbitrary spatial system, to observe the same two general conditions as in the case of equilibrium for an arbitrary plane system of forces.

For the equilibrium of a system of forces located at random in space, it is necessary and sufficient that both the principal vector $\mathbf{R}_{\text{pr}}$ *of the system and its principal moment* $\mathbf{M}_O$ *about an arbitrarily chosen centre of reduction be zero.*

These conditions may be cast in a more convenient (for practical purposes) analytical form. From the formulas for the magnitudes of the principal vector $\mathbf{R}_{\text{pr}}$ and the principal moment $\mathbf{M}_O$ of a spatial system of forces it follows that they vanish when the following equations hold:

$$\left.\begin{array}{lll} \sum X_k = 0, & \sum Y_k = 0, & \sum Z_k = 0, \\ \sum m_x(\mathbf{P}_k) = 0, & \sum m_y(\mathbf{P}_k) = 0, & \sum m_z(\mathbf{P}_k) = 0. \end{array}\right\} \quad (44)$$

For the equilibrium of a system of forces located at random in space, it is necessary and sufficient that the following two sets of sums be separately equal to zero: the sums of the projec-

tions of all forces on each of three arbitrarily chosen coordinate axes not lying in one plane, and the sums of the moments of all forces relative to each of three such axes. Since these equations of equilibrium of an arbitrary spatial system of forces are given without derivation, the following explanations will be of use.

The rotational action of any force about some axis is determined, as we know (Sec. 40), by the moment of the force about the given axis. The moment of a force about an axis is a scalar algebraic quantity, and the rotational action of a system of forces about any axis is determined by the algebraic sum of the moments of all forces of the system about that axis.

The fact that the algebraic sum of the moments of all forces of the system about some axis is equal to zero indicates that the given system of forces cannot impart rotation to the body about that axis. Thus the last three conditions of equilibrium,

$$\sum m_x(\mathbf{P}_k) = 0, \quad \sum m_y(\mathbf{P}_k) = 0 \quad \text{and} \quad \sum m_z(\mathbf{P}_k) = 0$$

signify that the system of forces under consideration cannot impart rotation to the body about any one of three coordinate axes not lying in one plane. Now this is equivalent to the fact that the given system of forces cannot impart any rotational motion at all to a free rigid body.

As for the first three equations of equilibrium, $\sum X_k = 0$, $\sum Y_k = 0$ and $\sum Z_k = 0$, they signify that the given system of forces cannot impart to a body translational* motion in the direction of any one of three coordinate axes not lying in the same plane. Now this in turn is equivalent to the fact that the given system of forces cannot impart to a free rigid body any translational motion at all.

Thus, if the six equations of equilibrium thus established are complied with, a system of forces located at random in space cannot impart to a free rigid body any translational motion and any rotational motion and for this reason it cannot alter the state of its motion (and the state of rest as a particular case) at all.

---

* In the case of the translational motion of a body (Sec. 64), all its points move in precisely the same way: in identical paths and with the same (at any instant) velocity.

Note that when setting up the equations of the moments there is no necessity to see that the axes about which the moments of the forces are taken should coincide with the axes of projections. For the sake of simplicity of solution of the equations, it is advisable to place the axis of projections perpendicularly to the line of action of one of the unknown forces, as a result of which the projections of that force will be eliminated from the appropriate projection equation. It is advisable to take the axis of moments in the plane of one of the unknown forces. Then the moment of that force about the given axis will be zero. In a word, the axes should always be chosen so that each of the six equations of equilibrium involve the smallest number of unknowns.

Problem 46. Referring to Fig. 112*a*, a rectangular door revolving about a vertical axis AB is opened at an angle $CAD = 60°$ and is held in that position by a load $Q = 160$ N suspended by a rope CD over a pulley with the end C attached to the door, and by a certain force **P** applied to point K of the door perpendicular to its plane. The door weighs $G = 480$ N, is 1.8 m wide ($AC = AD = BE = 1.8$ m) and 2.4 m high ($AB = 2.4$ m). Determine the magnitude of the force **P** and also the reactions of the hinge at point A and the step bearing at point B if $EK = 1.2$ m.

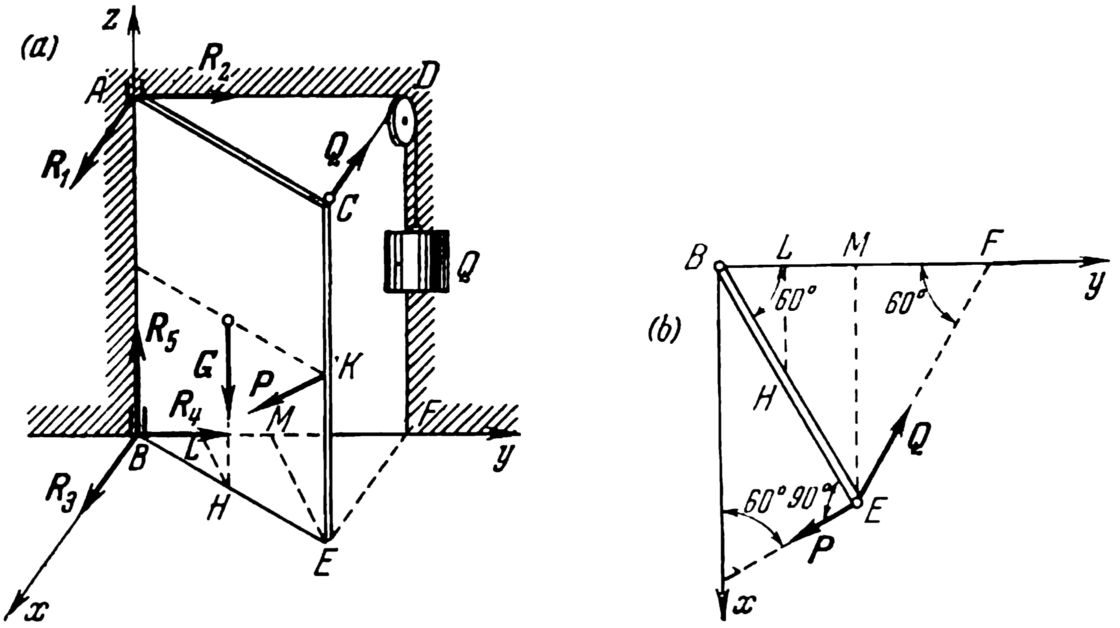


FIG. 112

Solution. The door is in equilibrium under the action of the active forces **G**, **Q**, **P** and the reactions of the bearing A and step bearing B. Draw the coordinate axes as shown in Fig. 112*a*, and decompose the reactions of constraints into components along these axes. Since the cylindrical hinge A admits sliding of the door in the vertical direction, its reaction has no vertical component and is resolved into only two components $\mathbf{R}_1$ and $\mathbf{R}_2$.

Now the reaction of the step bearing B yields the components $\mathbf{R}_3$, $\mathbf{R}_4$ and $\mathbf{R}_5$ in the directions of the three coordinate axes. The arrangement of the forces is shown in Fig. 112*a*. For the sake of convenience in determining the projections and moments of the forces **Q** and **P**, their projections on the plane xBy are shown in Fig. 112*b*.

We now set up a table of the projections of the forces on the chosen coordinate axes x, y, z and the moments of the forces about these axes.

Force	Projection of force on axes			Moment of force about axes		
	x	y	z	x	y	z
G	0	0	$-G$	$-G\times BL$	$G\times HL$	0
Q	$-Q\sin 60°$	$Q\cos 60°$	0	$-Q\cos 60°\times$ $\times AB$	$-Q\sin 60°\times$ $\times AB$	$Q\sin 60°\times BE$
P	$P\cos 60°$	$-P\sin 60°$	0	$P\sin 60°\times EK$	$P\cos 60°\times EK$	$-P\times BE$
$\mathbf{R}_1$	R_1	0	0	0	$R_1\times AB$	0
$\mathbf{R}_2$	0	R_2	0	$-R_2\times AB$	0	0
$\mathbf{R}_3$	R_3	0	0	0	0	0
$\mathbf{R}_4$	0	R_4	0	0	0	0
$\mathbf{R}_5$	0	0	R_5	0	0	0

The equilibrium equations take the form

$$\sum X_k = -Q\sin 60° + P\cos 60° + R_1 + R_3 = 0,$$

$$\sum Y_k = Q\cos 60° - P\sin 60° + R_2 + R_4 = 0,$$

$$\sum Z_k = -G + R_5 = 0,$$

$$\sum m_x(\mathbf{P}_k) = -G\times BL - Q\cos 60°\times AB + P\sin 60°\times EK - R_2\times AB = 0,$$

$$\sum m_y(\mathbf{P}_k) = G\times HL - Q\sin 60°\times AB + P\cos 60°\times EK + R_1\times AB = 0,$$

$$\sum m_z(\mathbf{P}_k) = Q\sin 60°\times BE - P\times BE = 0.$$

From Fig. 112*b* we find

$$BL = BH\cos 60° = \frac{BE}{2}\times\cos 60° = \frac{1.8}{2}\times\frac{1}{2} = 0.45\ \text{m},$$

$$HL = BH\sin 60° = \frac{1.8}{2}\times\frac{\sqrt{3}}{2} = 0.45\sqrt{3}\ \text{m}.$$

Substituting all data into the equations and solving them, we obtain

$$P = 80\sqrt{3}\ \text{N},\quad R_1 = -50\sqrt{3}\ \text{N},\quad R_2 = -110\ \text{N},$$
$$R_3 = 90\sqrt{3}\ \text{N},\quad R_4 = 150\ \text{N},\quad R_5 = 480\ \text{N}.$$

The negative values obtained for R_1 and R_2 show that the directions of these forces as indicated in Fig. 112 must be reversed.

Sec. 43. The equations of equilibrium of a spatial system of parallel forces

Using the general equations of equilibrium of forces located at random in space, we can find the equilibrium equations of a spatial system of parallel forces.

Suppose we have a system of parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_n$ (Fig. 113). Since the choice of coordinate axes is arbitrary, take the z-axis parallel to the given forces. Then set up six equilibrium equations of the arbitrary spatial system of forces. Since the axes x and y are perpendicular to the given parallel forces, the projections on these axes of each of the forces of the given system are zero. Hence, for such a choice of coordinate axes the equations $\sum X_k = 0$ and $\sum Y_k = 0$ are satisfied, irrespective of whether the system is in equilibrium or not, and for that reason cease to be conditions of equilibrium. Since all the given forces are parallel to the z-axis, their projections on that axis are equal to the magnitudes of these forces taken with the plus or minus sign, depending on their direction. Consequently, the equation $\sum Z_k = 0$ may be replaced by $\sum(\pm P_k) = 0$. Also, the equation $\sum m_z(\mathbf{P}_k) = 0$ is not needed since the moments of all forces about the z-axis parallel to them will always be equal to zero separately for any values of the forces and for any distances from the z-axis.

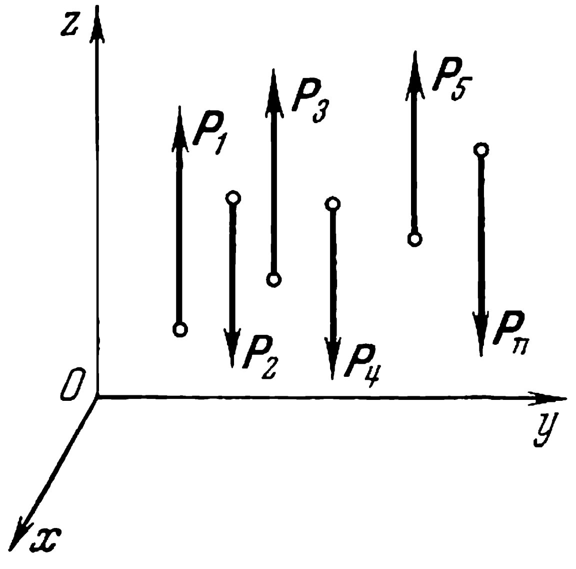


FIG. 113

Thus, only three equilibrium equations remain for the system of parallel forces:

$$\sum (\pm P_k) = 0, \quad \sum m_x (\mathbf{P}_k) = 0, \quad \sum m_y(\mathbf{P}_k) = 0. \quad (45)$$

For the equilibrium of a spatial system of parallel forces, it is necessary and sufficient that the following be equal to zero: the algebraic sum of all forces, the sums of the moments

System of forces	Number of independent equilibrium equations	Equations of equilibrium
Arbitrary spatial system of forces	6	$\sum X_k=0,\ \sum Y_k=0,\ \sum Z_k=0,$ $\sum m_x(\mathbf{P}_k)=0,\ \sum m_y(\mathbf{P}_k)=0,$ $\sum m_z(\mathbf{P}_k)=0$
Arbitrary plane system of forces	3	(1) $\sum X_k=0,\ \sum Y_k=0,\ \sum m_O(\mathbf{P}_k)=0,$ or (2) $\sum m_A(\mathbf{P}_k)=0,\ \sum m_B(\mathbf{P}_k)=0,$ $\sum m_C(\mathbf{P}_k)=0$ if the points A, B and C do not lie on a straight line, or (3) $\sum m_A(\mathbf{P}_k)=0,\ \sum m_B(\mathbf{P}_k)=0,$ $\sum X_k=0$ if the x-axis is not perpendicular to the straight line AB
Spatial system of parallel forces	3	$\sum P_k=0,\ \sum m_x(\mathbf{P}_k)=0,\ \sum m_y(\mathbf{P}_k)=$ $=0$ if the axes x and y lie in a plane perpendicular to the forces
Plane system of parallel forces	2	(1) $\sum P_k=0,\ \sum m_O(\mathbf{P}_k)=0,$ or (2) $\sum m_A(\mathbf{P}_k)=0,\ \sum m_B(\mathbf{P}_k)=0$ if the points A and B do not lie on a straight line parallel to the given forces
Spatial system of converging forces	3	$\sum X_k=0,\ \sum Y_k=0,\ \sum Z_k=0$
Plane system of converging forces	2	$\sum X_k=0,\ \sum Y_k=0$

of all forces about each of the two axes lying in a plane perpendicular to the given parallel forces.

An arbitrary system of forces in space—for the equilibrium of which the six equations established in Sec. 41 must be complied with—is a general case of the location of forces applied to a body. The equilibrium equations derived earlier for special cases of the location of forces could have been obtained from the given six equations just as it was done above for the spatial system of parallel forces.

For each case of the location of forces, a sufficient condition is the definite number of conditions of equilibrium, and therefore for each of them we can write only a definite number of independent equations of equilibrium. This is important to remember because if the number of unknowns exceeds the number of independent equations that may be set up for each given case of the location of forces, the problem becomes statically indeterminate.

The table given on p. 171 summarizes the above-established conditions of equilibrium for all locations of forces applied to a body.

Problem 47. Referring to Fig. 114, a load $G = 1$ kN lies on the platform of a three-wheel bogie truck at K. Find the pressure of each

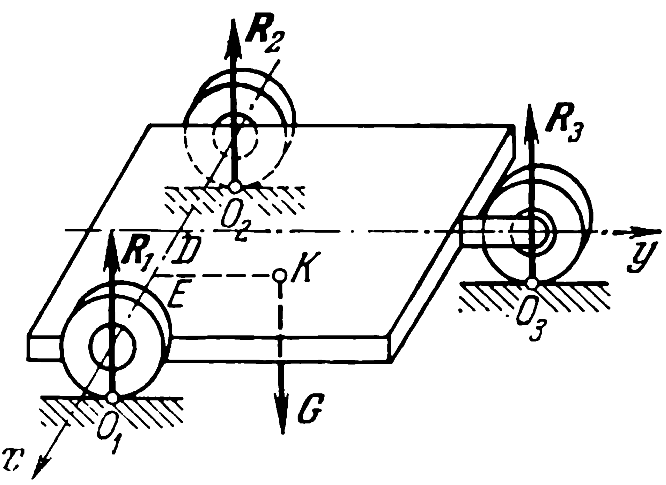


FIG. 114

wheel of the bogie truck on the floor (its own weight is disregarded) if $O_1O_2 = 1$ m, $O_3D = 1.6$ m, $O_1E = 0.4$ m and $EK = 0.6$ m. The point D is the midpoint of the line O_1O_2.

Solution. The bogie truck is in equilibrium under the action of a spatial system of parallel forces: the force of gravity **G** of the load and the reactions of the floor $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$, which are equal to the pressure exerted by the wheels on the floor. There are three unknowns and it is possible to set up three independent equations of equilibrium.

In the plane perpendicular to the lines of action of the given forces, take the axes x and y as shown in Fig. 114 and find the moments of all given forces about those axes.

Force	Moment of force about axes	
	x	y
G	$-G\times EK$	$G\times ED$
R_1	0	$-R_1\times O_1D$
R_2	0	$R_2\times O_2D$
R_3	$R_3\times O_3D$	0

The equilibrium equations are of the form

$$\sum P_k = G - R_1 - R_2 - R_3 = 0,$$

$$\sum m_x(\mathbf{P}_k) = -G \times EK + R_3 \times O_3D = 0,$$

$$\sum m_y(\mathbf{P}_k) = G \times ED - R_1 \times O_1D + R_2 \times O_2D = 0.$$

Substituting the given facts and solving the equations, we obtain

$$R_1 = 412.5\ \text{N}, \quad R_2 = 212.5\ \text{N}, \quad R_3 = 375\ \text{N}.$$

Obviously, the desired pressures of the wheels on the floor are equal in magnitude to the reactions just found.

Sec. 44. The equilibrium of a body having a fixed axis

All the above-established conditions of equilibrium are conditions of equilibrium of a free rigid body, that is, a body that is free to move in any manner in space.

In practical situations we ordinarily have to do with bodies that are not free, that is, bodies whose freedom of motion is limited by various kinds of constraints. By discarding the constraints and replacing their actions on the body by corresponding reactions (as we have done all along when solving problems) we may regard every nonfree body as a free body and, hence, write down for that body the appropriate equilibrium equations of a free body. If the constraints imposed on the body do not fix the body rigidly, the reactions of the constraints will not appear in all the equations of equilibrium of the body.

Equilibrium equations that do not contain constraint reactions show for what relationship between the active forces applied to the body and for what position of the body equilibrium is possible; and they *serve as conditions of equilibrium of a nonfree rigid body.*

The other equations of equilibrium merely serve to determine the unknown constraint reactions.

As an illustration for determining the conditions of equilibrium of a nonfree body let us consider the extremely important case of equilibrium of a body having two fixed points A and B or, what is the same thing, a fixed axis passing through two points. It is obvious that the only possible motion of the given body is that of rotation about its axis AB.

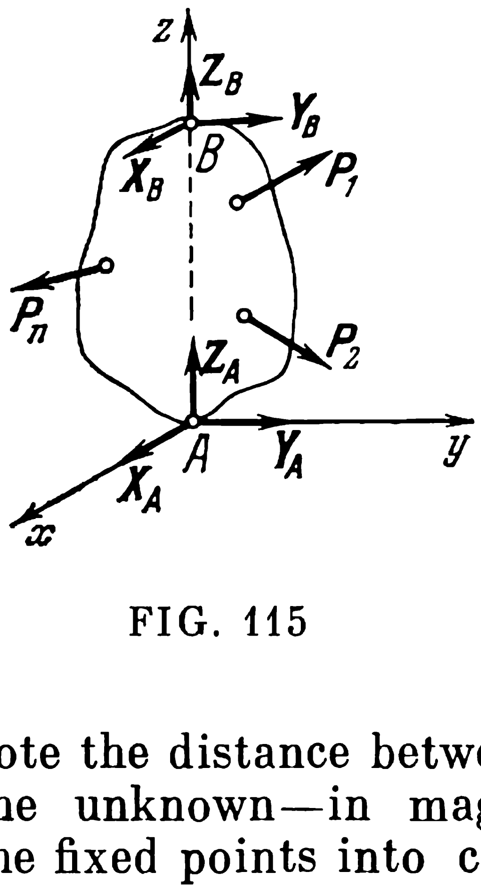


FIG. 115

Suppose an arbitrary spatial system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ is applied to a body (Fig. 115). Take point A, which is one of the fixed points of the body, as the coordinate origin and take the z-axis along the straight line AB. Denote the distance between the points A and B by a. Resolve the unknown—in magnitude and direction—reactions of the fixed points into components along the axes of coordinates. Denote these components by $\mathbf{X}_A$, $\mathbf{Y}_A$, $\mathbf{Z}_A$ and $\mathbf{X}_B$, $\mathbf{Y}_B$, $\mathbf{Z}_B$ respectively.

Replacing them by reactions, we can write six equilibrium equations for a free body under the action of a spatial system of forces applied to it:

$$
\begin{aligned}
&(1)\ \sum P_{kx}+X_A+X_B=0,\\
&(2)\ \sum P_{ky}+Y_A+Y_B=0,\\
&(3)\ \sum P_{kz}+Z_A+Z_B=0,\\
&(4)\ \sum m_x(\mathbf{P}_k)-aY_B=0,\\
&(5)\ \sum m_y(\mathbf{P}_k)+aX_B=0,\\
&(6)\ \sum m_z(\mathbf{P}_k)=0.
\end{aligned}
$$

In these equations, $\sum P_{kx}$, $\sum P_{ky}$, $\sum P_{kz}$ are the sums of the projections, on the appropriate axes, of all active forces applied to the body; $\sum m_x(\mathbf{P}_k)$, $\sum m_y(\mathbf{P}_k)$, $\sum m_z(\mathbf{P}_k)$ are the sums of the moments of these forces about the corresponding coordinate axes.

The sixth and last equilibrium equation is the only equation not involving the unknown reactions. Hence, the only condition that has to be satisfied by the active forces applied to the body is

$$\sum m_z(\mathbf{P}_k) = 0. \qquad (46)$$

For the equilibrium of a body having a fixed axis, it is necessary and sufficient that the sum of the moments of all active forces acting on the body about the fixed axis be equal to zero.

The remaining five equilibrium equations serve to determine the unknown reactions of the fixed points.

Since in the general case of forces located at random in space we have six unknown components of reactions, the problem of finding the reactions of two fixed points is, generally, a statically indeterminate problem.

However, if only one of the points is fixed, say A, and the other point, B, is permitted to slide freely (without friction) along the axis (as for example point A in problem 46), then the component $\mathbf{Z}_B$ vanishes and the problem becomes statically determinate.

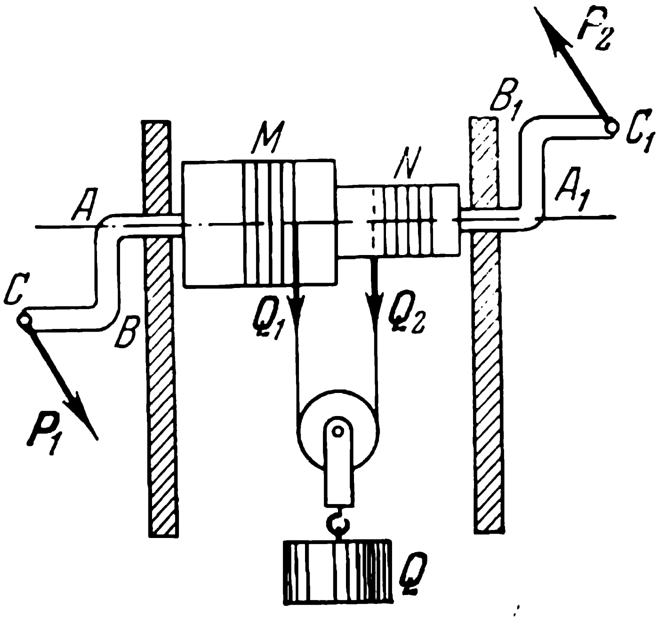


FIG. 116

Problem 48. Referring to Fig. 116, a differential windlass consists of a drum M of radius r_1 and a drum N of radius r_2 rotating on a common axle AA_1. Wound on these drums is a rope put through a mobile pulley to which is attached a load weighing Q. The rope is wound in such a way that when the crank of the windlass is turned, it winds off one drum and onto the other. Applied to handle BC is a force $\mathbf{P}_1$ perpendicular to the plane ABC. Applied to handle B_1C_1 is a force $\mathbf{P}_2$ perpendicular to the plane $A_1B_1C_1$ and equal in magnitude to force $\mathbf{P}_1$.

Find the relationship between the magnitudes P and Q under equilibrium if the length of the cranks AB and A_1B_1 is equal to l.

Solution. Disregarding the fact that the branches of rope that descend from drums M and N and go through the mobile pulley are not parallel, we can assume that the tension of each of the branches is equal to $Q_1 = Q_2 = Q/2$. Applying the condition of equilibrium of a body having a fixed axis (46), we obtain

$$\sum m_{AA_1}(\mathbf{P}_k) = P_1 l - Q_1 r_1 + Q_2 r_2 + P_2 l = 0.$$

From this, taking into account that $P_1 = P_2 = P$ and $Q_1 = Q_2 = Q/2$, we obtain

$$P = \frac{r_1 - r_2}{4l} Q.$$

Problem 49. Referring to Fig. 117*a*, two belt pulleys are mounted on a transmission shaft. The diameters of the pulleys are $D_1 = 40$ cm, $D_2 = 50$ cm; the distances between the pulleys and bearing A are $a = 1$ m and $b = 3$ m; the distance between the bearings A and B is $l = 4$ m. The branches of the belt on the first pulley form an angle $\alpha = 20°$ with the vertical; the branches of the belt on the second pulley are horizontal. Given: the tensions $T_1 = 2$ kN and $T_2 = 4$ kN of the branches of the first belt, and the tension $T_3 = 5$ kN of the upper branch of the second belt. It is required to find under what tension T_4 of the lower branch of the second belt the shaft, acted upon by the applied forces, will be in equilibrium, and also to determine the reactions of the bearings generated by the tensions of the belts.

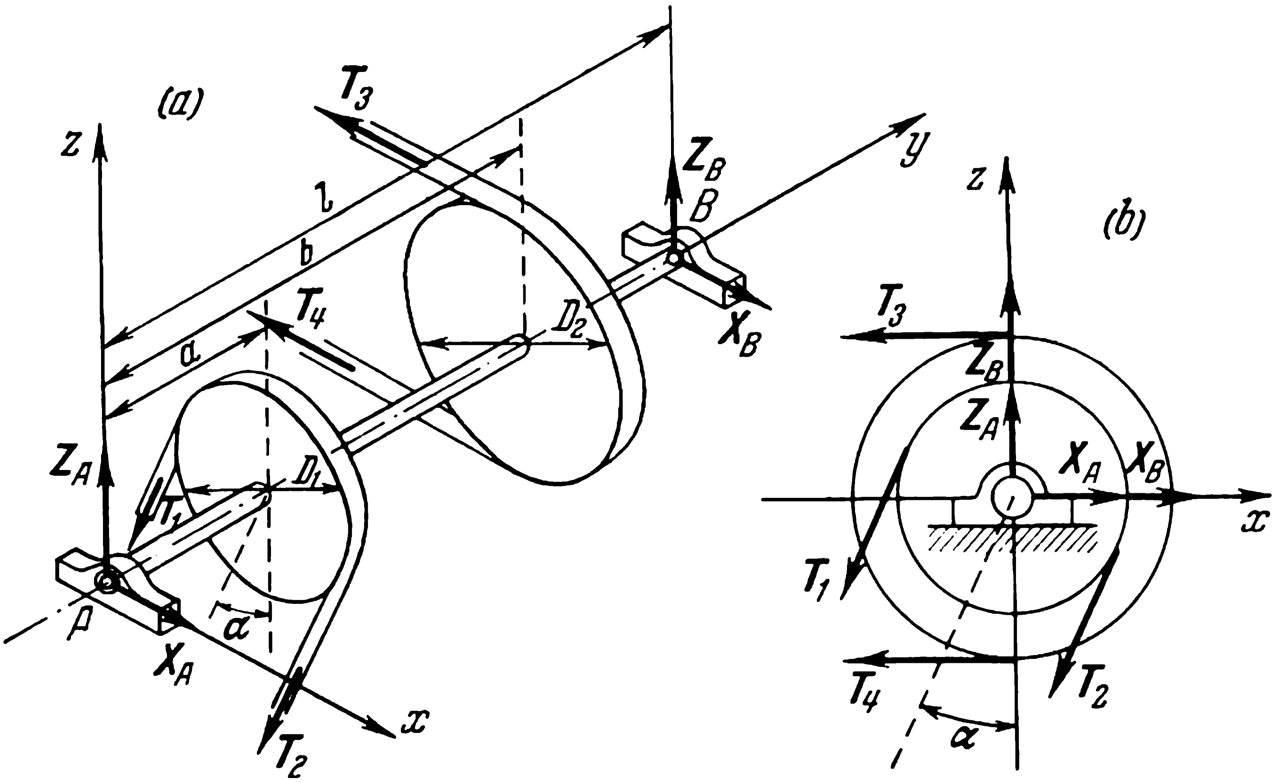


FIG. 117

Solution. The tension T_4 is immediately determined from the condition of equilibrium of a body having a fixed axis:

$$\sum m_y(\mathbf{P}_k) = -T_3\frac{D_2}{2} + T_4\frac{D_2}{2} - T_1\frac{D_1}{2} + T_2\frac{D_1}{2} = 0.$$

Solving this equation, we get

$$T_4 = \frac{T_3D_2 + (T_1 - T_2)D_1}{D_2} = \frac{5\times 50 + (2-4)\times 40}{50} = 3.4 \text{ kN}.$$

Since all forces are located in planes perpendicular to the axis of the shaft, the reactions of the bearings will not have components directed along the axis of the shaft, that is, components along the y-axis. Denote the components of the desired reactions of the bearings A and B along the axes x and z by $\mathbf{X}_A$, $\mathbf{Z}_A$ and $\mathbf{X}_B$, $\mathbf{Z}_B$ respectively. To determine them, we project all forces applied to the shaft (Fig. 117*b*) on the x- and z-axis and find their moments about these axes (see table).

Force	Projection of force on axes		Moment of force about axes	
	x	z	x	z
$\mathbf{T}_1$	$-T_1\sin\alpha$	$-T_1\cos\alpha$	$-T_1\cos\alpha\times a$	$T_1\sin\alpha\times a$
$\mathbf{T}_2$	$-T_2\sin\alpha$	$-T_2\cos\alpha$	$-T_2\cos\alpha\times a$	$T_2\sin\alpha\times a$
$\mathbf{T}_3$	$-T_3$	0	0	T_3b
$\mathbf{T}_4$	$-T_4$	0	0	T_4b
$\mathbf{X}_A$	X_A	0	0	0
$\mathbf{Z}_A$	0	Z_A	0	0
$\mathbf{X}_B$	X_B	0	0	$-X_Bl$
$\mathbf{Z}_B$	0	Z_B	Z_Bl	0

Setting up the appropriate equations of equilibrium, we obtain

$$\sum X_k = -T_1\sin\alpha - T_2\sin\alpha - T_3 - T_4 + X_A + X_B = 0,$$
$$\sum Z_k = -T_1\cos\alpha - T_2\cos\alpha + Z_A + Z_B = 0,$$
$$\sum m_x(\mathbf{P}_k) = -aT_1\cos\alpha - aT_2\cos\alpha + lZ_B = 0,$$
$$\sum m_z(\mathbf{P}_k) = aT_1\sin\alpha + aT_2\sin\alpha + bT_3 + bT_4 - lX_B = 0.$$

Solving these equations, we find

$X_B \approx 6.8$ kN, $X_A \approx 3.6$ kN, $Z_B \approx 1.4$ kN, $Z_A \approx 4.2$ kN.

CHAPTER VIII

THE CENTRE OF PARALLEL FORCES AND THE CENTRE OF GRAVITY OF A BODY

Sec. 45. The centre of parallel forces

Knowing the rules for the composition of two parallel forces, it is easy, via successive composition, to find the resultant for any system of parallel forces. For example, suppose three* parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ (Fig. 118) all in the same direction are applied to a body at points A_1, A_2, and A_3. By first combining the two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ via the appropriate rule, we obtain their resultant $\mathbf{R}_{12}$. Then combining by the same rule the force $\mathbf{R}_{12}$ with $\mathbf{P}_3$, we obtain the resultant $\mathbf{R}_P$ of all three given forces. This resul-

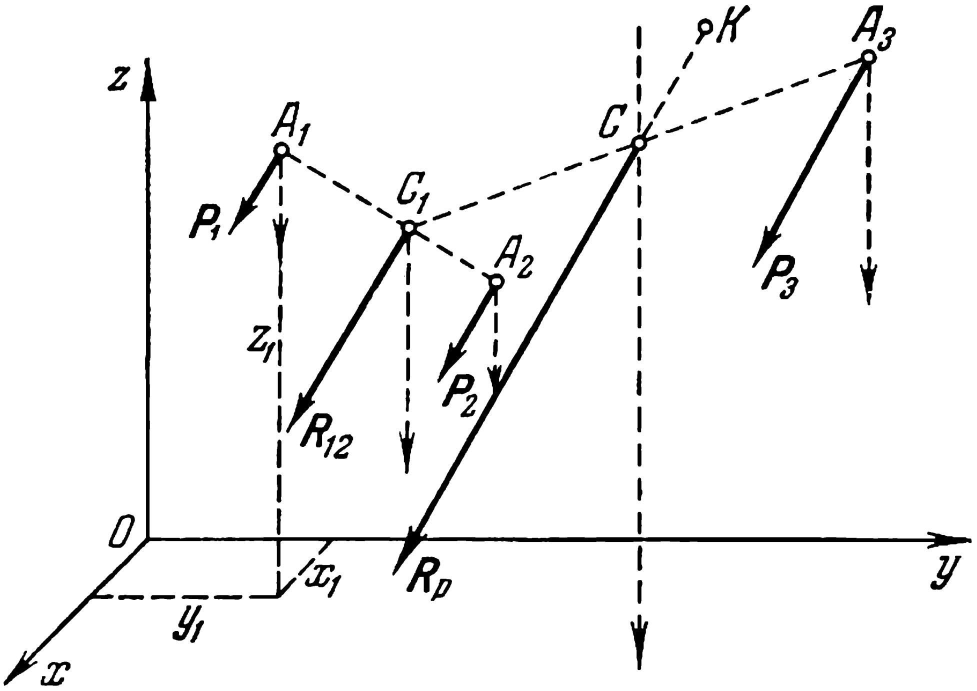


FIG. 118

---

* Three forces are taken for the sake of simplicity, but the method described here can be extended to any number of parallel forces.

tant is clearly parallel to the given forces and is in the same direction. The magnitude of the resultant is equal to the sum of the magnitudes of the component forces:

$$R_P = R_{12} + P_3 = P_1 + P_2 + P_3 = \sum P_k.$$

It now remains to determine the position of the point C through which the line of action of the resultant passes. For the point of application of the resultant we can of course take any point lying on its line of action; however, it turns out that only one of them, precisely point C, which is determined by means of a successive combining of forces, has a very important and special property.

This property consists in the fact that if we turn all the given forces about their points of application through the same angle without changing their parallel character, the line of action of their resultant, having been rotated through the same angle (as shown in Fig. 118 by the dashed lines), will again pass through the point C.

Now, if the point of application of the resultant is carried to some other point beforehand on the line of its action, say, to point K, then after a rotation of the given forces, the line of action of their resultant will not pass through that point.

The point C is termed the centre of a system of parallel forces.

From the foregoing it follows that the *centre of a given system of parallel forces is a point through which the line of action of their resultant passes under any rotation of the forces of the system about their points of application through the same angle and in the same direction.*

Now let us derive a formula for determining the coordinates of the centre of a system of parallel forces. Let us take a spatial system of coordinate axes and denote the coordinates of the points of application of the given forces: A_1 by x_1, y_1, z_1, A_2 by x_2, y_2, z_2, and A_3 by x_3, y_3, z_3.

We denote the coordinates of the centre of parallel forces C by x_C, y_C, z_C. We first compute the abscissa x' of point C_1 to which is applied the resultant $\mathbf{R}_{12}$ of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$. For this purpose we take advantage of the familiar formula of analytic geometry for the coordinates of a point that divides a line segment in a given ratio m/n:

$$x' = \frac{x_1 + \frac{m}{n} x_2}{1 + \frac{m}{n}}$$

where x_1 and x_2 are the coordinates of the end points of the given line.

Since C_1 divides the distance between the points of application of the component forces into parts that are inversely proportional to these forces, we have $A_1C_1/C_1A_2 = P_2/P_1$ and in our case $m/n = P_2/P_1$.

Hence

$$x' = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}.$$

We now find the abscissa x_C of point C at which the resultant $\mathbf{R}_P$ of the given forces is applied, that is, the abscissa of the centre of three parallel forces.

Since

$$\frac{C_1C}{CA_3} = \frac{P_3}{R_{12}} = \frac{P_3}{P_1 + P_2},$$

it follows here that $m/n = P_3/(P_1 + P_2)$. By the same formula we get

$$x_C = \frac{(P_1 + P_2)\, x' + P_3x_3}{P_1 + P_2 + P_3}$$

or, substituting the value of x' found above,

$$x_C = \frac{(P_1 + P_2)\, \frac{P_1x_1 + P_2x_2}{P_1 + P_2} + P_3x_3}{P_1 + P_2 + P_3}$$

$$= \frac{P_1x_1 + P_2x_2 + P_3x_3}{P_1 + P_2 + P_3} = \frac{\sum P_kx_k}{\sum P_k}.$$

In similar fashion we can find the other coordinates of the centre of parallel forces as well. We thus obtain the following formulas for the coordinates of the centre of a system of parallel forces:

$$\left.\begin{aligned} x_C &= \frac{\sum P_k x_k}{\sum P_k}, \\ y_C &= \frac{\sum P_k y_k}{\sum P_k}, \\ z_C &= \frac{\sum P_k z_k}{\sum P_k}. \end{aligned}\right\} \tag{47}$$

Formulas (47) for the coordinates of the centre of parallel forces hold true for any number of parallel forces. If the system involves forces of the opposite direction, and $\sum P_k \neq 0$, then by P_k one is to understand the algebraic value of the force, that is, its magnitude taken with the plus sign for one direction and with the minus sign for the opposite direction of the force. Thus, in these formulas, the symbol $\sum$ is to be understood as an algebraic sum: in the numerator we have the sum of the products of the algebraic value of each force into the appropriate coordinate of its point of application, and in the denominator the algebraic sum of all forces.

Sec. 46. The centre of gravity of a body

By the law of universal gravitation, all particles of a body near the earth's surface are acted upon by forces attracting them to the earth, that is by forces of gravity*. These forces are directed along a radius of the earth and intersect approximately at the centre of the earth. But since the distance to the centre of the earth is very great in comparison to the distances between the particles of a body of ordinary dimensions, we can say, with a very high degree of accuracy, that the forces of gravity of all the separate particles of a body are parallel*.

---

* The force of gravity of a particle is, to be precise, the resultant of two forces: the force of attraction of that particle to the earth and the centrifugal force due to the revolution of the particle together with the earth. This latter force is greatest when the particle is located on the equator (that is, when it is farthest from the axis of rotation). But even in this case the centrifugal force of the particle is roughly 300 times less than the force of its attraction to the earth.

The resultant of the forces of gravity of all the separate particles of a body is termed the force of gravity of the body; the magnitude of this force is said to be the weight of the body.

No matter how we turn a body or alter its position in space, the forces of gravity of its separate particles remain parallel to each other (vertical); relative to the body, they will turn about their points of application, all the while retaining their parallelism and magnitude. Now in such rotation, the resultant of parallel forces always passes through one and the same point, the centre of the given system of parallel forces. From this fact it follows that the centre of gravity lies at a very definite point for every body and does not alter its position relative to that body when the body itself changes its position.

The centre of gravity of a body is a point invariably fixed in the body through which the line of action of the force of gravity passes for any position of the body in space.

It must be noted that it is not correct to imagine the centre of gravity as a material particle of the body to which the force of gravity of the body is applied directly. The force of gravity is the resultant, which in problems of mechanics (and not even in all problems), may conditionally replace the action on the body of the forces of gravity that are exerted on the elementary particles of the body. The centre of gravity of a body is only a point through which the line of action of that resultant always passes. The centre of gravity of a body may lie at a point where there are no material particles belonging to the body at all.

For example, the centre of gravity of a homogeneous sphere with a concentric hollow is the geometric centre for the reason that, no matter what position the sphere is in, the resultant of the forces of gravity of its elementary particles passes through that point.

---

* If the distance between particles is 31 m, the angle between their corresponding verticals constitutes only one second.

Sec. 47. The coordinates of the centre of gravity of a body. The static moment of the area of a plane figure

Since the centre of gravity of a body is the centre of the parallel forces of gravity of its particles, its coordinates are determined by formulas similar to the formulas (47):

$$\left.\begin{aligned} x_C &= \frac{\sum G_k x_k}{\sum G_k} = \frac{\sum G_k x_k}{G}, \\ y_C &= \frac{\sum G_k y_k}{\sum G_k} = \frac{\sum G_k y_k}{G}, \\ z_C &= \frac{\sum G_k z_k}{\sum G_k} = \frac{\sum G_k z_k}{G}, \end{aligned}\right\} \qquad (48)$$

where x_C, y_C, z_C are the coordinates of the centre of gravity of the body, G_k is the weight of an arbitrary particle of the body, x_k, y_k, z_k are the coordinates of that particle, $\sum G_k x_k$, $\sum G_k y_k$, $\sum G_k z_k$ are the sums made up of the products of the weight of each particle of the body into the appropriate coordinate of that particle, and G is the weight of the whole body.

Let us consider a special case of a homogeneous body, which is a body of the same density at all its points. Denoting the volume of some one particle of the body by v_k, the volume of the whole body by v, and the weight of unit volume of the body by γ, we obtain the weight of one particle of the body, $G_k = v_k\gamma$, and the weight of the whole body, $G = v\gamma$. Substituting these values into the preceding formula, we obtain

$$x_C = \frac{\sum G_k x_k}{G} = \frac{\sum v_k \gamma x_k}{v\gamma} = \frac{\gamma \sum v_k x_k}{v\gamma} = \frac{\sum v_k x_k}{v}.$$

Similar formulas can clearly be obtained for the other two coordinates as well. As is evident from the preceding expression, the coordinates of the centre of gravity of a homogeneous body do not depend on the constant γ, which characterizes the substance of the given body, but depend solely on the volume occupied by the body and on its shape.

That is why the centre of gravity of a homogeneous body is called the *centre of gravity of a volume.* Its coordinates are

$$x_C=\frac{\sum v_k x_k}{v},\quad y_C=\frac{\sum v_k y_k}{v},\quad z_C=\frac{\sum v_k z_k}{v}. \quad (49)$$

In these formulas, x_k, y_k and z_k are the coordinates of a particle of the body.

In practical situations, it is frequently required to determine the position of the centre of gravity of plane figures. These figures may be pictured as thin homogeneous plates whose thickness may be ignored. The volumes of the separate particles of such a plate are proportional to the areas of the corresponding elements of the figure, and the coordinates of its centre of gravity will depend solely on the area of the figure and its shape.

For this reason the centre of gravity of a homogeneous thin plate of constant thickness having the configuration of a plane figure is called the *centre of gravity of the area* of the given plane figure.

Its coordinates are

$$x_C=\frac{\sum F_k x_k}{F},\quad y_C=\frac{\sum F_k y_k}{F}, \quad (50)$$

where F_k is the area of an arbitrary element of the figure, x_k, y_k are the coordinates of that element, F is the area of the entire figure.

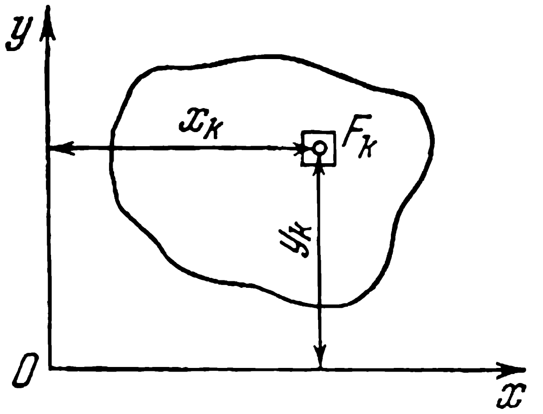


FIG. 119

The formulas (50) may be modified somewhat. Take an arbitrary figure (Fig. 119) and partition its area into separate elements F_k. The product of the area of an element of the figure into the shortest distance* of its centre of gravity to some axis lying in the same plane is termed the *static moment of the element of the figure about that given axis.*

---

* This distance is taken to be positive on one side of the axis and negative on the other side.

The sum of the static moments of all separate elements into which the given area is partitioned, if they are all taken about some one axis, is called the *static moment of the area of the given figure about that axis.*

Denoting the static moments of the area of the figure about the coordinate axes x and y by S_x and S_y respectively, we have

$$S_x = \sum F_k y_k, \quad S_y = \sum F_k x_k. \tag{51}$$

The static moment of area* obviously has the dimensions of the cube of the length (cm^3, mm^3 and so forth) because it is a product of the area measured in units of length squared into the distance measured in units of length to the first power. Replacing the numerators of formulas (50) by customary symbols for the static moments of area, we obtain different expressions for the coordinates of the centre of gravity of a plane figure:

$$x_C = \frac{S_y}{F}, \quad y_C = \frac{S_x}{F}. \tag{52}$$

These formulas can be used to compute the coordinates of the centre of gravity of a plane figure if we know its static moments about the coordinate axes, and, conversely, we can compute the static moments if the position of the centre of gravity of the figure is known.

If the area of the cross-section of a homogeneous body is the same throughout its length and the cross-sectional dimensions are very small compared to the length, then such a body (say, a figure made of wire) may be regarded as a material line. The weights and volumes of the separate parts of such a body will be proportional to their lengths, and the coordinates of the centre of gravity will depend solely on the length and shape of the line.

The centre of gravity of a homogeneous body whose cross-sectional area is the same throughout its length and is small compared to the length is termed the *centre of gravity of a line.*

---

* Static moments of area find application in the theory of strength of materials.

Its coordinates are

$$x_C = \frac{\sum l_k x_k}{l}, \quad y_C = \frac{\sum l_k y_k}{l}, \quad z_C = \frac{\sum l_k z_k}{l}, \qquad (53)$$

where l_k is an element of length of the body, x_k, y_k, z_k are the coordinates of that element, and l is the length of the whole body.

Sec. 48. The centre of gravity of a symmetric body

Theorem. *If a homogeneous body has a plane, an axis, or a centre of symmetry, the centre of gravity of the body lies, respectively, in the plane, on the axis or in the centre of symmetry.*

Proof. If the given body is symmetric about some plane (Fig. 120), then each particle on one side of that plane is associated with a particle of equal weight located symmetrically on the other side of the plane.

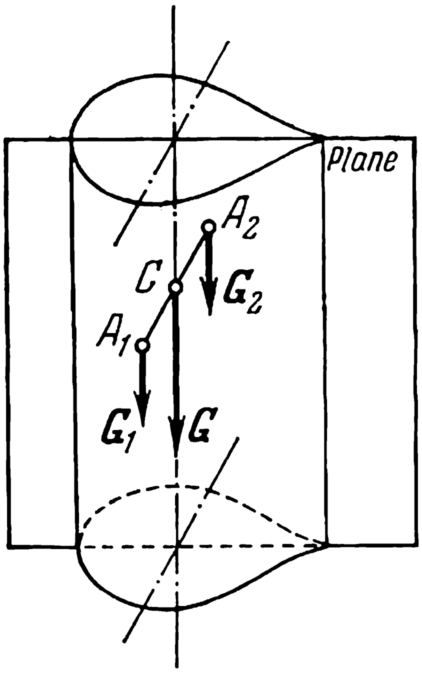


FIG. 120

Take some particle A_1 on one side of the plane and find a symmetrical particle A_2 on the other side.

These particles will be acted upon by forces of gravity $\mathbf{G}_1$ and $\mathbf{G}_2$ that have the same magnitude. The resultant $\mathbf{G}$ of these two equal and parallel forces will be applied to the midpoint of the line A_1A_2, that is, in the plane of symmetry.

By thus combining the weights of each pair of symmetrical particles, we obtain a system of parallel forces lying in the plane of symmetry of the body. Clearly, the centre of the given system of parallel forces (which is the centre of gravity of the body) will lie in the same plane.

The proof is the same for cases where the body has an axis or centre of symmetry.

Corollary 1. *The centre of gravity of a line segment of a material homogeneous straight line lies at its midpoint.*

Corollary 2. *The centre of gravity of the area of a homogeneous parallelogram lies at the point of intersection of its diagonals, which point is the centre of symmetry of the parallelogram.*

Corollary 3. *The centres of gravity of the areas of a regular polygon, a circle, an ellipse and the volume of a sphere lie at their geometric centres.*

Sec. 49. The position of the centre of gravity of certain homogeneous bodies of simple shape

1. The centre of gravity of the area of a triangle

Partition the area of a triangle (Fig. 121) by lines parallel to the base AB into a large number of very narrow strips that may be regarded as segments of a material straight line. The centre of gravity of each such segment lies at its midpoint; from this we conclude that the centre of gravity of the whole area of the triangle lies somewhere on the line joining the midpoints of these line segments, that is on the median DE of triangle ABD.

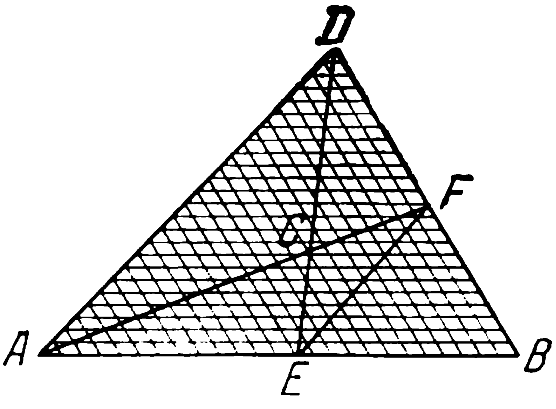


FIG. 121

Partitioning the area of the triangle by straight lines parallel to one of the sides, say BD, and reasoning in similar fashion, we arrive at the conclusion that the centre of gravity of the area of the triangle must lie on the median AF. Hence it lies *at the point of intersection of the medians of the triangle.*

Proof is given in geometry* that the point of intersection of the medians of a triangle divides each of them in the ratio 1 : 2, that is, the point lies on a median at a distance equal to one third of the median from the point of intersection of median and the appropriate side. We can therefore say that the *centre of gravity of the area of a triangle lies on its median at a distance one third of the median from the point of intersection of the median and the appropriate side of the triangle.*

---

* This is quite evident if we consider the similarity of the triangles EFC and ACD and the triangles BEF and ABD.

To determine the centre of gravity of the area of an arbitrary polygon, partition it into triangles and determine their centres of gravity. Assuming the weight of each triangle as being applied at its centre of gravity, we then find the centre of the system of parallel forces thus obtained.

2. The centre of gravity of an arc of a circle

Referring to Fig. 122, suppose we have an arc AB of a circle of radius R with central angle 2α. Through the midpoint of the arc draw a radius OD. Since the radius OD is the axis of symmetry of the arc, the desired centre of gravity C lies somewhere on that radius. Take the x-axis along OD and take the centre O of the arc as the coordinate origin. We then only have to find one coordinate of the centre of gravity, x_C. By formula (53), the coordinate x_C may be expressed thus:

$$x_C = \frac{\sum l_k x_k}{l},$$

where l_k is an element of arc of the circle, x_k is its coordinate, and l is the total arc length.

Let us partition arc AB into a large number of small elements l_k and then draw the radius OM_k to the midpoint of one of them. From an end point of the element drop a perpendicular M_kN_k on the x-axis. On the element of arc, construct a right triangle $a_kb_kc_k$ whose legs are parallel to the x-axis and to the chord AB (the arc a_kc_k may be taken to be rectilinear due to its smallness).

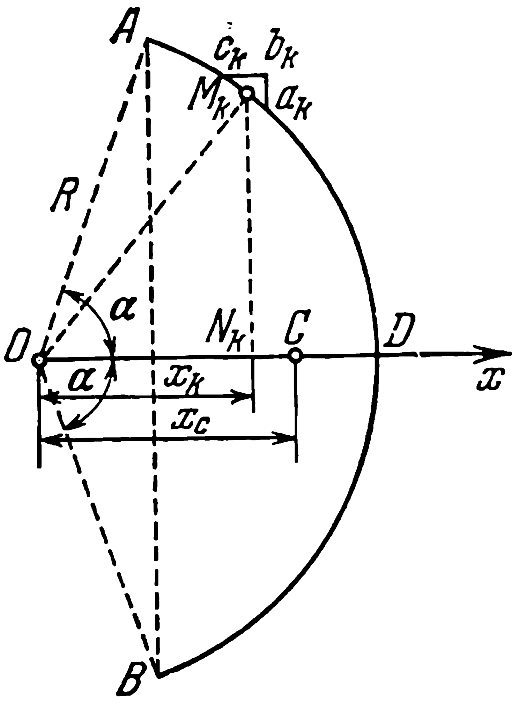


FIG. 122

Due to perpendicularity of the sides, the triangle $a_kb_kc_k$ is similar to the triangle OM_kN_k.

Hence

$$\frac{a_kc_k}{OM_k}=\frac{a_kb_k}{ON_k}.$$

In this equality, $a_kc_k = l_k$, $OM_k = R$ and ON_k is the coordinate of the midpoint of a_kc_k, that is, the coordinate x_k of the element l_k of arc.

Substituting these values into the above-found relationship, we obtain $l_k/R = a_kb_k/x_k$ or $l_kx_k = Ra_kb_k$. Performing the same constructions for all elements of arc AB, we obtain similar equalities for each of them. By summing these equalities termwise and taking into account that the sum of the legs a_kb_k of the triangles constructed for all elements of the arc is equal to the chord AB, we have

$$\sum l_kx_k = \sum Ra_kb_k = R\sum a_kb_k = R \times AB.$$

Substituting this value of $\sum l_kx_k$ into the formula for the coordinate of the centre of gravity, we obtain

$$x_C=\frac{\sum l_kx_k}{l}=\frac{R\times AB}{l}.$$

From Fig. 122, we have $AB/2 = AO \sin \alpha$, or

$$AB = 2R \sin \alpha.$$

Now the arc length $\overset{\frown}{AB}$ is, as we know, equal to the radius multiplied by the central angle expressed in radians. Consequently $l = 2\alpha R$. Substituting the values of AB and l thus found into the expression for x_C and simplifying, we finally get

$$x_C=R\frac{\sin \alpha}{\alpha}, \tag{54}$$

where R is the radius of the arc, α is half the central angle (expressed in radians) corresponding to the given arc.

3. The centre of gravity of the area of a circular sector

Suppose we have a circular sector $OADB$ of radius R with central angle 2α (Fig. 123).

Take the centre of the circle, O, for the coordinate origin and *take the* x-*axis along the radius* OD *drawn through the midpoint of arc* AB. Partition the given sector into n equal elementary sectors, that is, sectors with very small central angles. Because of the smallness of the arcs bounding these sectors, the latter may be regarded as isosceles triangles whose centres of gravity lie on the arc of a circle of radius $OA_1 = r = = \frac{2}{3}R$ (the median of an elementary triangle coincides with the radius drawn to the midpoint of the appropriate elementary arc) at equal distances from one another. The weights of the elementary sectors applied to these centres are equal (this is because their areas are equal). The problem clearly reduces to determining the centre of equal parallel forces whose points of application are uniformly distributed along the arc A_1B_1 of the circle, that is to say, in the limit, as the number of elementary sectors n increases without bound, it reduces to determining the centre of gravity of the homogeneous arc A_1B_1. As was established earlier, its centre of gravity lies on the axis of symmetry Ox at a distance

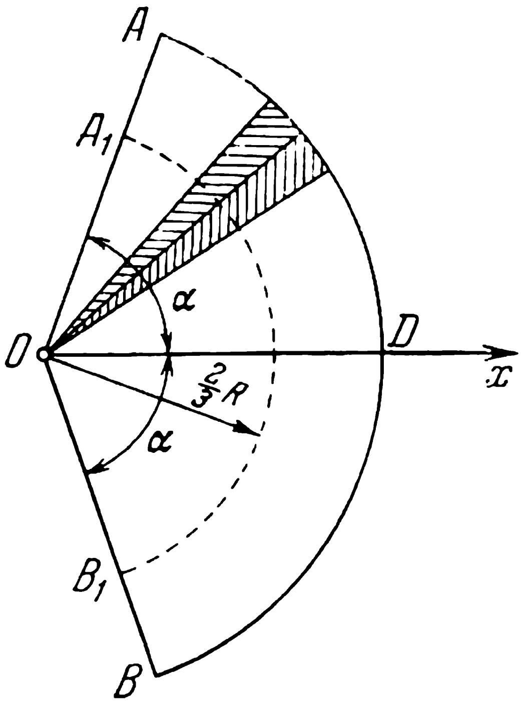


FIG. 123

$$x_C = r\frac{\sin\alpha}{\alpha}$$

from the centre of the circle.

In this case, the radius of the arc $r = OA_1 = \frac{2}{3}R$ and consequently the desired coordinate of the centre of gravity of the area of a circular sector is given by the formula

$$x_C = \frac{2}{3} R \frac{\sin \alpha}{\alpha}. \tag{55}$$

4. The centre of gravity of the volume of a prism

Mentally partition the prism by planes parallel to the base into a large number of very thin lamina of the same thickness. Since the thickness is infinitesimal, the lamina may be regarded as flat polygons whose centres of gravity lie on one straight line O_1O_2 joining the centres of gravity of the upper and lower bases of the prism. The weights of the polygons applied to the centres are equal due to the equality of the areas. Thus the problem reduces to determining the centre of equal parallel forces whose points of application are uniformly distributed along the straight line O_1O_2, that is, in the limit, when the number of partitions increases without bound, it reduces to determining the centre of gravity of the homogeneous line segment O_1O_2. From this we conclude that the centre of gravity of the volume of a prism lies at the midpoint of the line segment joining the centres of gravity of its upper and lower bases. Since a cylinder may be regarded as a prism with an infinite number of lateral faces, the centre of gravity of a homogeneous cylinder is determined by the same rule as that used for a prism.

Sec. 50. Determining the position of the centre of gravity of figures and bodies of intricate shape

1. Analytical method

In order to determine the position of the centre of gravity of figures of a complicated geometric shape, the first thing is to partition them into simpler shapes (if of course this is possible) for which the positions of the centres of gravity are known, and then to determine the position of the centre of gravity of the whole figure or solid by means of appropriate formulas established in Sec. 47; in these formulas, v_k, F_k and l_k are to be understood as the volume, area and length of the parts into which the given solid, figure or line is partitioned, and x_k, y_k, z_k are the coordinates of the centres of gravity of the parts.

If the figures or bodies are not homogeneous, that is, if they consist of parts of different density, then we separate them into homogeneous parts and multiply the volumes, areas and lengths of the parts [that appear in formulas (49), (50) and (53)] into the specific weight that corresponds to each part. If the given body or figure has hollows or apertures, the same devices and formulas are used in determining the centre of gravity of such a body or figure, but the volumes and areas of the eliminated parts are assumed to be negative.

Widely used in engineering practice is rolled steel stock* of various structural shape, that is, of diverse cross-sectional shape. The shapes of these cross-sections and their dimensions are set up by USSR State Standards (so-called GOST). Tables of what is known as a normal assortment of rolled stock can be found in reference works and include for each size of the appropriate structural shape all the necessary information, such as the geometric dimensions of the section, the area of the section, and the coordinates of the centre of gravity. Using these data and the procedures described above, one can determine the position of the centre of gravity of a composite section obtained by combining several standard structural shapes.

Problem 50. Referring to Fig. 124, find the position of the centre of gravity of a symmetric bar truss *ADBE* with dimensions: $AB = 6$ m, $DE = 3$ m, and $EF = 1$ m.

Solution. Since the truss is symmetric, its centre of gravity lies on the axis of symmetry *DF*. For the chosen system of coordinate axes (Fig. 124), the abscissa of the centre of gravity of the truss is

$$x_C = AF = \frac{AB}{2} = 3 \text{ m}.$$

---

* Rolled stock is obtained in steel plants by reducing the cross-sectional area of a workpiece put through the rotating rolls of a steel mill.

We thus have to find the ordinate y_C of the centre of gravity of the truss. To determine it, partition the truss into separate parts (bars). Their lengths are found from the appropriate triangles. From $\triangle AEF$ we have $AE = BE = \sqrt{AF^2 + FE^2} = \sqrt{3^2 + 1^2} = 3.17$ m. From $\triangle ADF$ we have $AD = \sqrt{AF^2 + (DE + EF)^2} = \sqrt{3^2 + 4^2} = 5$ m. The centre of gravity of each bar lies at the midpoint of the bar. The ordinates of these centres are easily found from the drawing (Fig. 124).

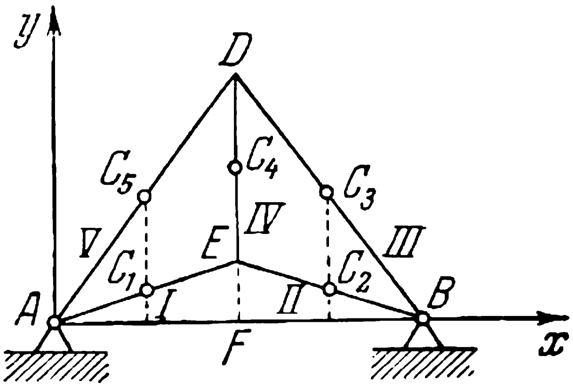


FIG. 124

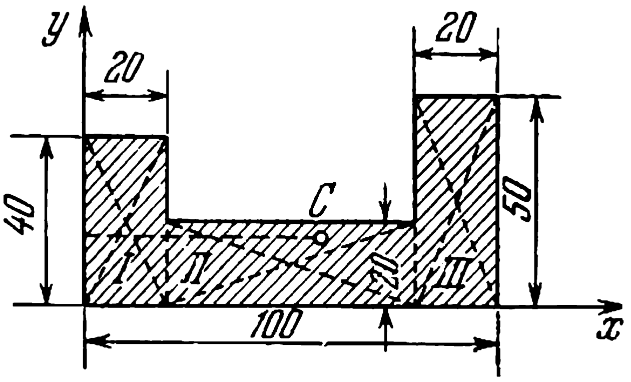


FIG. 125

We tabulate the lengths and ordinates of the centres of gravity of the separate parts of the truss and then, via formula (53), we determine the ordinate y_C of the centre of gravity of the given plane truss.

Part of truss	Length l_k of part (in metres)	Ordinate y_k of centre of gravity of each part (in metres)
I	3.17	0.5
II	3.17	0.5
III	5	2
IV	3	2.5
V	3	2

$$y_C = \frac{\sum l_k y_k}{l}$$

$$= \frac{3.17 \times 0.5 + 3.17 \times 0.5 + 5 \times 2 + 3 \times 2.5 + 5 \times 2}{3.16 + 3.16 + 5 + 3 + 5} \approx 1.54 \text{ m}.$$

Hence, the centre of gravity C of the whole truss lies on the axis DF of symmetry of the truss at a distance of 1.54 m from point F.

Problem 51. Find the static moments, about the coordinate axes, of the area of a sheet whose dimensions (in centimetres) are given in Fig. 125, and find the coordinates of the centre of gravity of that area.

Solution. Divide the given area into three rectangles. The centre of gravity of each rectangle lies at the intersection of its diagonals. The coordinates of these centres, just as the areas of the rectangles, are readily determined from the drawing. We first set up a table and find the static moments of the area by means of formulas (51) and then we find the coordinates of its centre of gravity by means of formulas (52):

Part of area	Area F_k of each part (in cm²)	Coordinates of centre of gravity of each part (in cm)	
		x_k	y_k
I	800	10	20
II	1200	50	10
III	1000	90	25

$$S_x = \sum F_k y_k = 800\times 20 + 1200\times 10 + 1000\times 25 = 53\,000 \text{ cm}^3,$$

$$S_y = \sum F_k x_k = 800\times 10 + 1200\times 50 + 1000\times 90 = 158\,000 \text{ cm}^3,$$

$$x_C = \frac{S_y}{F} = \frac{158\,000}{800+1200+1000} = \frac{158\,000}{3000} \approx 52.7 \text{ cm},$$

$$y_C = \frac{S_x}{F} = \frac{53\,000}{3000} \approx 17.7 \text{ cm}.$$

Problem 52. Find the centre of gravity C of the area of a circular segment ADB of radius $R = 30$ cm if the angle $AOB = 2\alpha = 60°$ (Fig. 126).

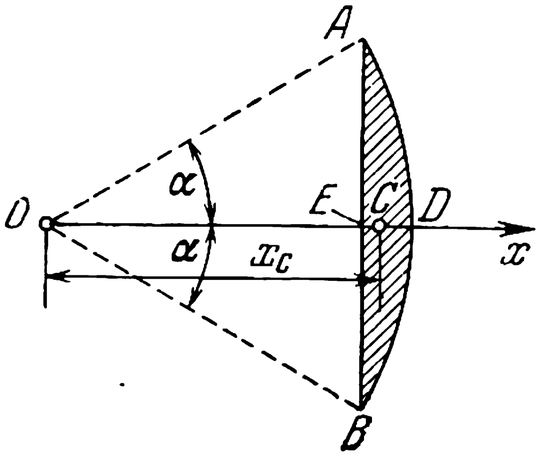


FIG. 126

Solution. The desired centre of gravity C lies on the axis of symmetry passing through the centre of the circle O and the midpoint D

of arc AB. We assume this axis to be the x-axis and O to be the origin. We will consider the given circular segment as consisting of two figures: a circular sector $OADB$ and a triangle AOB, the triangular area being regarded as negative.

The area of the circular sector is $F_1=\frac{\pi r^2}{360}\times 60=\frac{\pi r^2}{6}=$ $=\frac{3.14\times 30^2}{6}=471$ cm². The abscissa of its centre of gravity, $x_1=$ $=\frac{2}{3}R\frac{\sin\alpha}{\alpha}=\frac{2}{3}R\frac{\sin 30^\circ}{\pi/6}=\frac{4\times 30\times 0.5}{3.14}\approx 19.1$ cm. The area of the triangle, $F_2=\frac{AB\times OE}{2}=AE\times OE=R\sin\alpha\times R\cos\alpha=\frac{R^2\sin 2\alpha}{2}=$ $=\frac{30^2\times 0.866}{2}\approx 390$ cm². The abscissa of its centre of gravity, $x_2=\frac{2}{3}OE=\frac{2}{3}R\cos\alpha=\frac{2}{3}\times 30\times 0.866\approx 17.3$ cm.

Using formula (50) we determine the abscissa of the centre of gravity of the given circular segment:

$$x_C=\frac{\sum F_k x_k}{F}=\frac{F_1x_1+(-F_2x_2)}{F_1+(-F_2)}=\frac{471\times 19.1-390\times 17.3}{471-390}$$

$$\approx 27.8 \text{ cm.}$$

Problem 53. A body (Fig. 127) consists of a wooden cylinder (the radius of the base of which is r = 5 cm, height l = 30 cm) and of two

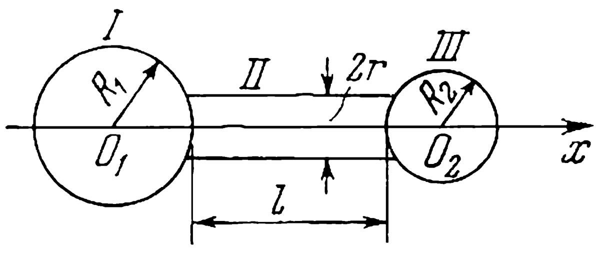


FIG. 127

attached steel balls with radii R_1 = 12 cm and R_2 = 9 cm. Determine the position of the centre of gravity of the body if the specific gravity of wood is $\gamma_1 = 0.6$ and that of steel is $\gamma_2 = 7.85$.

Solution. The desired centre of gravity lies on the axis of symmetry passing through the centres of the balls O_1 and O_2. We take the origin of coordinates at the centre O_1 of the larger ball, and for the

x-axis we take the axis of symmetry. We then partition the body into three parts and set up a table of volumes and coordinates (abscissas) of the centres of gravity.

Part of body	Volume v_k of part (cm³)	Specific gravity γ_k	Abscissa x_k of centre of gravity of part (cm)
I	$\frac{4}{3}\pi\times 12^3=2300\pi$	7.85	0
II	$\pi\times 5^2\times 30=750\pi$	0.6	27
III	$\frac{4}{3}\pi\times 9^3=972\pi$	7.85	51

To find the abscissa of the centre of gravity of the entire nonhomogeneous body, we take advantage of formula (48):

$$x_C=\frac{\sum G_k x_k}{G}=\frac{\sum \gamma_k v_k x_k}{\sum \gamma_k v_k}$$

$$=\frac{7.85\times 2300\pi\times 0+0.6\times 750\pi\times 27+7.85\times 972\pi\times 51}{7.85\times 2300\pi+0.6\times 750\pi+7.85\times 972\pi}=15.4 \text{ cm}.$$

Problem 54. Find the static moments about the coordinate axes and find the position of the centre of gravity of a cross-section (Fig. 128*a*) made up of an equal-leg angle 100 × 100 × 10, a channel bar No. 24, and a sheet 180 × 10.

Solution. From normal assortment tables for rolled steel* we take the following data:

I. Equal-leg angle (Fig. 128*b*). GOST 8509-57. Section No. 10. Width of leg $b=100$ mm. Thickness of leg $d=10$ mm. Area of cross-section $F=19.2$ cm². Distance of centre of gravity from bases of legs $z_0=2.83$ cm.

II. Channel bar (Fig. 128*c*). GOST 8240-56. Section No. 24. Height of wall $h=240$ mm. Width of leg $b=90$ mm. Thickness of wall $d=5.6$ mm. Area of cross-section $F=30.6$ cm². Distance of centre

---

* The data given here are taken from 1965 tables, although USSR State Standards for 1972 are now used.

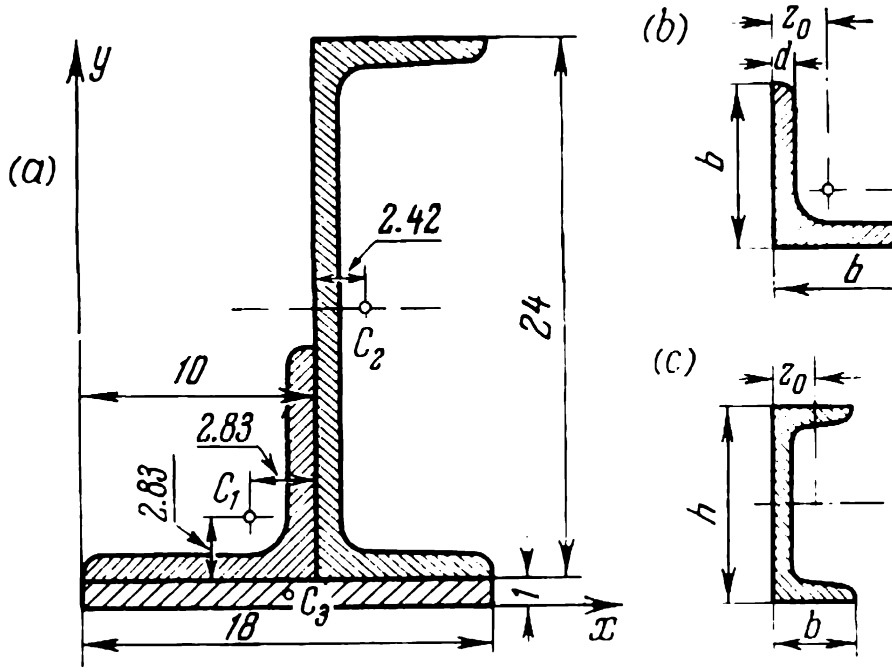


FIG. 128

of gravity from outer edge of vertical wall $z_0 = 2.42$ cm. (The channel bar has a horizontal axis of symmetry and, hence, its centre of gravity lies on that axis.)

III. Strip steel. GOST 103-57. Rectangular cross-section. Width of strip 180 mm. Thickness 10 mm. Area of cross-section $F = 18$ cm².

Let us number the separate parts of the section on the basis of the data given above and then state the dimensions (in cm) in Fig. 128*a*. The axes are those indicated in the drawing.

Part of cross-section	Area of each part (cm²)	Coordinates of centre of gravity of each part (cm)	
		x_k	y_k
I	19.2	$10 - 2.83 = 7.17$	$1 + 2.83 = 3.83$
II	30.6	$10 + 2.42 = 12.42$	$1 + 12 = 13$
III	18	9	0.5

The static moment of the cross-section about the y-axis:

$$S_y = \sum F_k x_k = 19.2 \times 7.17 + 30.6 \times 12.42 + 18 \times 9 = 679.6 \text{ cm}^3.$$

The static moment of the cross-section about the x-axis:

$$S_x = \sum F_k y_k = 19.2 \times 3.83 + 30.6 \times 13 + 18 \times 0.5 = 462.3 \text{ cm}^3.$$

The coordinates of the centre of gravity of the cross-section:

$$x_C = \frac{S_y}{F} = \frac{679.6}{19.2+30.6+18} = \frac{679.6}{67.8} \approx 10.0 \text{ cm},$$

$$y_C = \frac{S_x}{F} = \frac{462.3}{67.8} \approx 6.8 \text{ cm}.$$

2. *Experimental method*

Weighing is one of the possible experimental methods for finding the position of the centre of gravity of a body. To illustrate, suppose it is required to find the position of the centre of gravity of a connecting rod. Suspend the connecting rod at A by a rope from a fixed point O and at point B let it rest on a platform of a decimal balance (Fig. 129) so that the rod is in a horizontal position. Now suppose the force of pressure of the rod on the platform (and hence the reaction **R** of the platform applied to the rod) found by weighing equals P in absolute value.

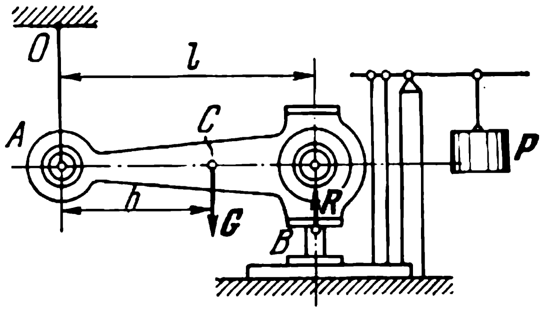


FIG. 129

If we know the weight G of the connecting rod and the distance l between points A and B, it is easy to find the distance h from A to the centre C of gravity of the rod.

According to the equation of equilibrium,

$$\sum m_A(\mathbf{P}_k) = Gh - Rl = 0, \quad \text{whence} \quad h = \frac{Rl}{G} = \frac{Pl}{G}.$$

CHAPTER IX

STABILITY OF EQUILIBRIUM

Sec. 51. The concept of stability of equilibrium of a body having a point of support or an axis of rotation

Under actual conditions, a body in equilibrium is acted upon by all kinds of transient shocks, which are usually called "perturbations". They are capable of imparting minute deviations from the position of equilibrium of the body thus upsetting the conditions of equilibrium of the forces applied to the body.

A body is said to be in stable equilibrium if after any small deviation from the position of equilibrium the forces applied to the body tend to return the body to its original position.

Contrariwise, the *equilibrium of a body is said to be unstable if as a result of some small deviation from the original position of equilibrium the body does not return to that position.*

An example of stable equilibrium is the equilibrium of a ball in contact with a concave surface at its lowest point (Fig. 130*a*); an instance of unstable equilibrium is the equilibrium of a ball in contact with a convex surface at its upper point (Fig. 130*b*).

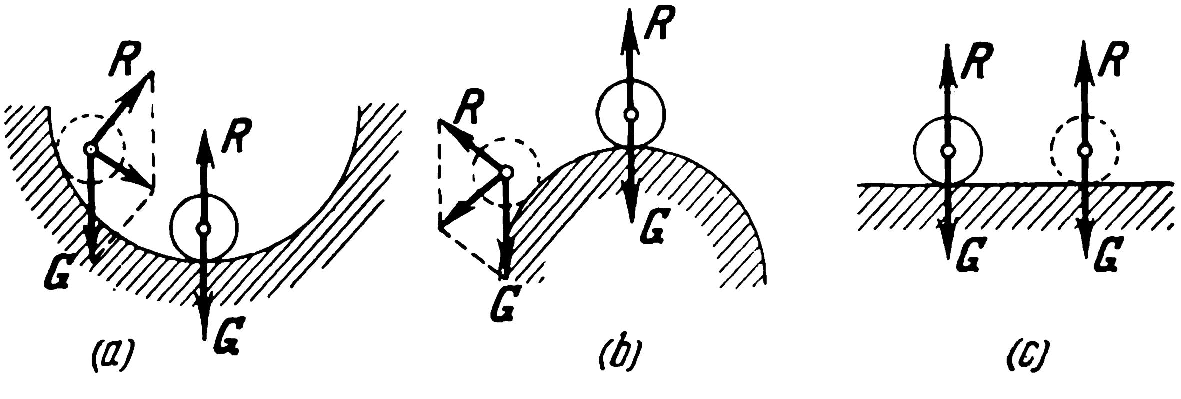


FIG. 130

If the surface on which the ball rests is smooth and, hence, friction between the ball and the surface may be disregarded, then the reaction of the surface will be directed along the normal to the ball surface at the appropriate point.

In both cases, the balls will be in equilibrium since the forces **G** and **R** applied to them are equal in magnitude and opposite in direction. But the equilibrium of the ball depicted in Fig. 130*a* is stable equilibrium, whereas that shown in Fig. 130*b* is unstable. Indeed, if we deflect the balls from their positions of equilibrium, the forces **G** and **R** will no longer be directed along a single straight line and their resultant (now not equal to zero) will tend to return the ball to its original position in the first instance and to remove it still farther from that position in the second case.

If the ball rests on a horizontal plane (Fig. 130*c*), the forces **G** and **R** applied to it remain balanced for any displacement of the ball. This *equilibrium, which is retained for small displacements of the body from the original position, is called indifferent equilibrium.*

A similar relationship between the position of the centre of gravity of the body and its type of equilibrium also exists for a body having a fixed horizontal axis of rotation.

If (see Fig. 131) the centre of gravity of a body rises when the body is deflected from its position of equilibrium,

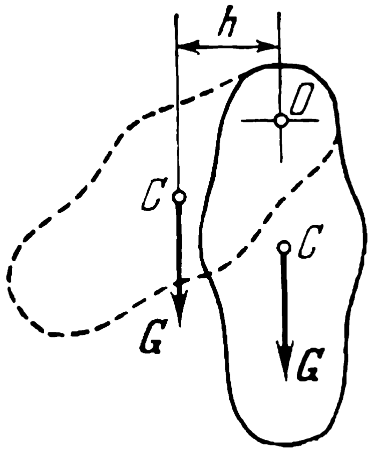


FIG. 131

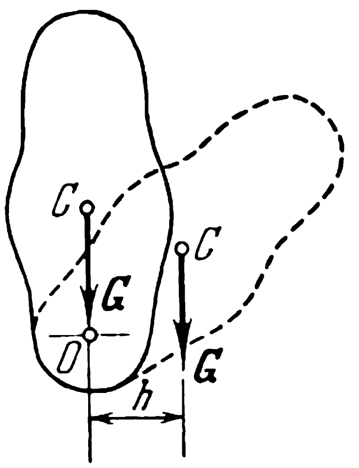


FIG. 132

the force **G** of gravity yields a moment about the fixed axis passing through point O that is numerically equal to Gh and has a direction such that it returns the body to the original position of equilibrium.

Contrariwise, if the position of the axis of rotation about the centre of gravity of the body is such that its centre of gravity drops when the body is deflected from its position of equilibrium (Fig. 132), then the force G of gravity yields a moment about the axis of rotation that is numerically equal to *Gh* and has a direction such that removes the body from the original position of equilibrium.

Finally, if the centre of gravity of the body, for any deflection of the body, remains unchanged (this occurs when the centre of gravity of the body lies on the axis of rotation of the body), then the force of gravity of the body will not, under these deflections, yield any moment about the axis of rotation.

From the foregoing we can draw the following conclusion: *the equilibrium of a body having a point of support or a horizontal axis of rotation is stable when its centre of gravity occupies the lowest of all possible adjacent positions, it is unstable when the centre of gravity occupies the highest of such positions, and it is indifferent when the height of its centre of gravity for all positions of the body remains unchanged.*

Sec. 52. The stability of a body resting on a plane

Picture a body with its base resting on a horizontal plane (Fig. 133*a*). Turn the body about its edge *A* to position *II*

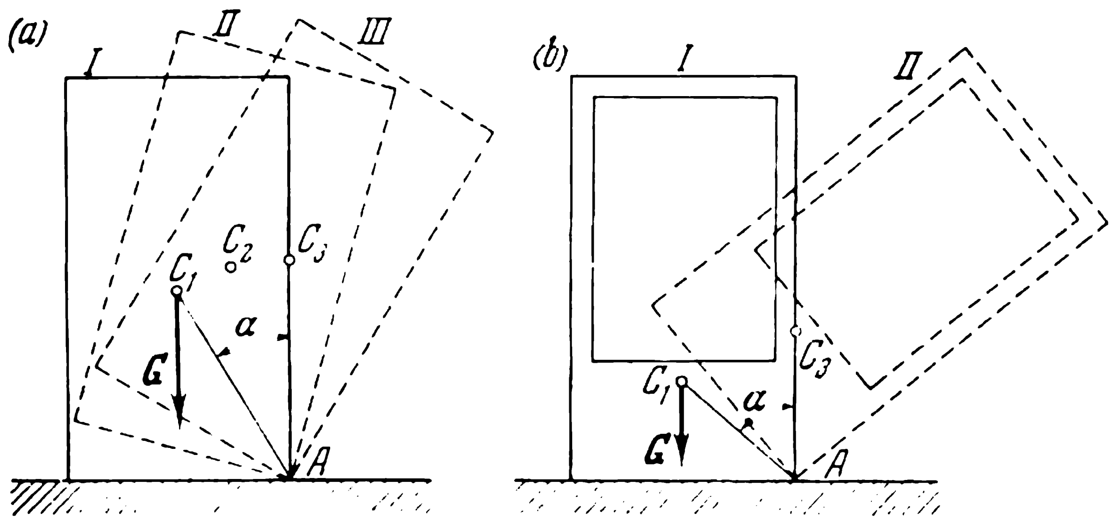


FIG. 133

so that the line of force G (the force of gravity of the body) remains on the same side of edge *A*. It is clear that in that position the force of gravity G yields a moment about the axis of rotation that tends to return the body to its original position of equilibrium. If we turn the body to position *III*, that is, so that the line of the force G intersects the axis of support, then the moment of the force of gravity about the axis will be zero and the body will have an equal probability of returning to its original position or of falling onto its right-hand face. This is a position of unstable equilibrium because only the slightest force is required to deflect the body from that position, thus upsetting the equilibrium.

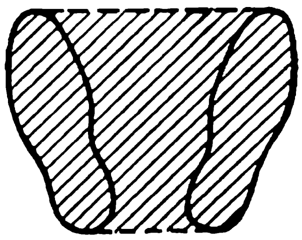


FIG. 134

By comparison, position *I* is a position of stable equilibrium since slight deviations of the body from that position return the body to its original position.

The angle of turn of the body from its stable position to an unstable position is termed the angle of stability. This angle α is the greater, the broader the base of the body and the lower its centre of gravity. For instance, if we retain position *I* of the body shown in Fig. 133 but artificially lower its centre of gravity (by making its lower part heavier than the upper portion), the angle of stability α will increase. To overturn the body in that case (see Fig. 133*b*) requires rotating it through a larger angle.

The ability of a body to return to its original position of equilibrium after the forces upsetting equilibrium cease to act is termed dynamic stability of the body.

From the foregoing it follows that the *dynamic stability of a body increases with increasing width of the support area of the body and with a lowering of its centre of gravity.*

If a body does not rest on a continuous surface but on several points not lying on one straight line, the support area is that area formed by lines joining all the points. For example, for a human being, the support area is that shown cross-hatched in Fig. 134. To increase dynamic stability in a vehicle, for instance, the wheels of the vehicle are put as far apart as possible and the load is arranged so that the heavier objects are put below and the lighter ones on top. With the same purpose in mind, objects are made with massive bases in order to lower the centre of gravity.

In practical situations, one frequently has to do not only with ensuring dynamic stability of bodies (as, for example, various vehicles, river and ocean vessels), that is, not only with ensuring the return to an original position of equilibrium after small deviations from that position, but with establishing no deviations at all from equilibrium.

The ability of a body to oppose any, even the slightest, deviation from equilibrium is termed static stability of the body.

Suppose a body of weight **G** is acted upon by a force **P** that tends to overturn it about edge *B* (in Fig. 135 is depicted a section *ABCD* of the body by a vertical plane passing through its centre of gravity and the line of action of the force **P**). The body will be in equilibrium only if the line of action of the resultant $\mathbf{R}_P$ of the forces **G** and **P** acting on the body passes to the left of point *B* (Fig. 135*a*), that is, if it intersects the plane of support inside the outline of the base of the body. Now, if the line of action of the resultant $\mathbf{R}_P$ of the forces **G** and **P** (Fig. 135*b*) passes to the right of point *B*, that is, outside the area of support, then it will yield a moment that tends to separate the base of the body from the support plane and overturn it about point *B*. In the limiting case of equilibrium, when the line of action of the resultant $\mathbf{R}_P$ of the forces G and **P** passes through *B*, its moment about that point will be zero. But by Varignon's theorem, the moment of the resultant $\mathbf{R}_P$ is equal to the sum of the moments of the component forces. Consequently, in the limiting case of equilibrium of the body, $\sum m_B(\mathbf{P}_k) = Ga - Ph = 0$ or $Ga = Ph$. The product Ga, that is, the *product of the weight of the body by its arm about a possible axis of rotation of the body is called the moment of stability of the body.*

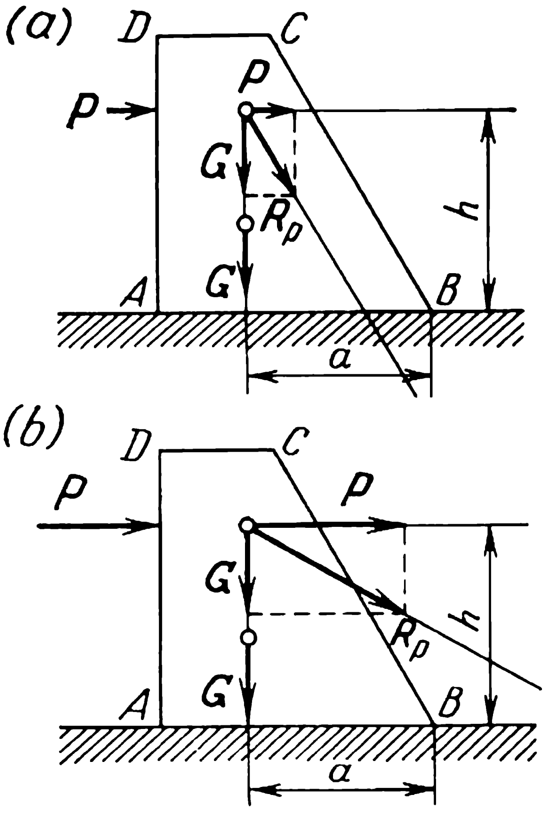


FIG. 135

The product Ph, that is, the *product of the magnitude of the overturning force by its arm about a possible axis of rotation of the body is termed the overturning moment.*

If $Ga < Ph$, the body will turn over; but if $Ga > Ph$, it will remain in equilibrium. Thus, *for static stability of a body, it is necessary that the moment of stability be greater than the overturning moment. The ratio of the stability moment to the overturning moment is called the coefficient of stability*:

$$k = \frac{Ga}{Ph}. \tag{56}$$

To ensure stability of structures, this coefficient must always exceed unity. Its values for various structures are indicated by the USSR State Standard (GOST). Ordinarily, k lies between 1.5 and 2.

Problem 55. Find the coefficient of stability of a brick column (Fig. 136) if its width $a = 1$ m, its thickness $b = 0.7$ m, height $h = 4$ m, and wind pressure $q = 1$ kPa $= 1$ kN/m^2 normal to the surface of the column. The masonry has a specific weight of $\gamma = 24$ kN/m^3.

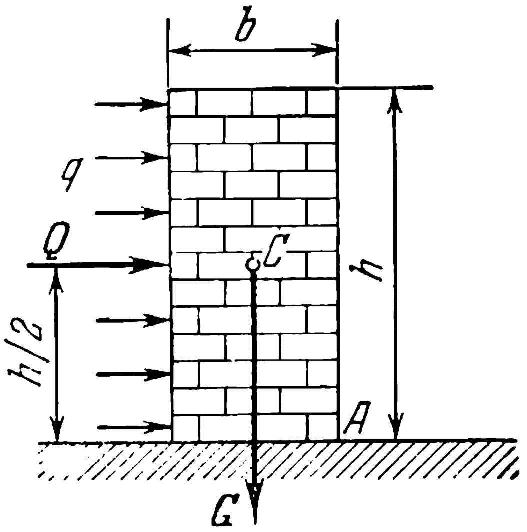


FIG. 136

Solution. Suppose the wind blows from the left. The area exposed to wind pressure is $F = ah = 1 \times 4 = 4$ m^2. The line of action of the resultant of the wind pressure will pass through the centre of gravity C of the column and is equal in magnitude to $Q = qF = 1 \times 4 = 4$ kN.

This force tends to overturn the column to the right, turning it about edge A. The overturning moment of this force about edge A is

$$m_A(Q) = Q\frac{h}{2} = 4 \times \frac{4}{2} = 8 \text{ kN}\cdot\text{m}.$$

The column weighs

$$G = abh\gamma = 1 \times 0.7 \times 4 \times 24 = 67.2 \text{ kN}.$$

The moment of stability of the column about edge A is

$$m_A(\mathbf{G}) = G\frac{b}{2} = 67.2 \times \frac{0.7}{2} \approx 23.5 \text{ kN}\cdot\text{m}.$$

The coefficient of stability of the column $k = \frac{m_A(\mathbf{G})}{m_A(\mathbf{Q})} = \frac{23.5}{8} \approx$ ≈ 2.94.

Problem 56. A hoisting crane is placed on a stone foundation (Fig. 137). The weight of the crane is $G_1 = 40$ kN and is applied to its centre of gravity C at a distance $b = 0.75$ m from the axis of the crane.

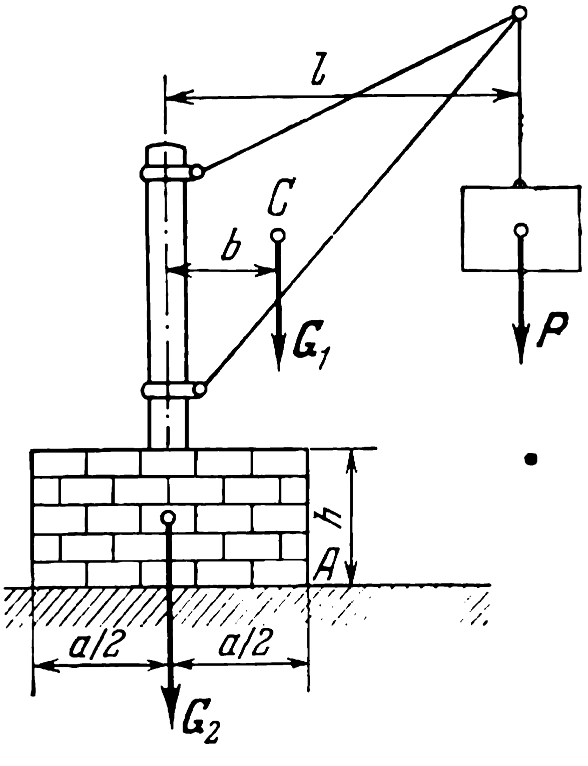


FIG. 137

The boom of the crane is $l = 4$ m. The foundation has a square base with side $a = 2$ m. The specific weight of the material of the foundation is $\gamma = 20$ kN/m^3. Calculate the necessary height of the foundation h if the crane has a lifting capacity of $P = 25$ kN and the coefficient of stability of the crane must be $k = 2$.

Solution. The force **P** tends to overturn the crane (together with the foundation) about the edge A. Its overturning moment about that edge is

$$m_A(\mathbf{P}) = P\left(l - \frac{a}{2}\right).$$

The moment of stability of the crane is equal to the sum of the moments about edge A of two forces: the force of gravity of the crane $\mathbf{G}_1$ and the force of gravity of the masonry $\mathbf{G}_2$:

$$\sum m_A(\mathbf{G}) = G_1\left(\frac{a}{2}-b\right)+G_2\frac{a}{2},$$

where the weight of the masonry of the foundation $G_2 = a^2h\gamma$.

For the given coefficient of stability $k = 2$, the moment of stability of the crane must be equal to

$$\sum m_A(\mathbf{G}) = km_A(\mathbf{P}),$$

whence

$$G_1\left(\frac{a}{2}-b\right)+a^2h\gamma\frac{a}{2}=kP\left(l-\frac{a}{2}\right).$$

Solving this equation, we obtain

$$h=\frac{2kP\left(l-\frac{a}{2}\right)-2G_1\left(\frac{a}{2}-b\right)}{a^3\gamma}$$

$$=\frac{2\times 2\times 25\,(4-1)-2\times 40\,(1-0.75)}{2^3\times 2}=1.75 \text{ m}.$$

PART II

KINEMATICS

CHAPTER X

INTRODUCTION TO KINEMATICS

Sec. 53. The subject and essentials of kinematics

Kinematics is the section of theoretical mechanics that treats of the motions of bodies geometrically, quite apart from factors giving rise to such motion.

Like geometry that treats of the spatial properties of bodies and discards all other material features (weight, hardness and the like), kinematics treats of the motions of bodies as a process, developing in time, of the continual variation of their positions in space, and likewise disregards the connection of such motion with the material structure of the bodies and with the forces acting on them. Consequently, in kinematics a moving body is regarded merely as a certain geometric image.

Kinematics is based entirely on the axioms and propositions of geometry but differs in that it considers both the space traversed by a moving body and the time during which the motion is performed.

Kinematics is very important not only with respect to investigations into the last section of theoretical mechanics, dynamics, but also for studying the geometric properties of motion of the parts of various types of mechanisms. Progress in engineering and the problems of designing complicated mechanisms and machines during the first half of the 19th century led to the development of kinematics as an independent part of theoretical mechanics. And at present, too, kinematics is developing in terms of its application to the designing and investigating of mechanisms and machines.

Space and time, which enter into the concept of motion, are forms of the existence of matter, without which neither matter itself nor its motion is conceivable. As Lenin wrote in *Materialism and Empiriocriticism*, "There is nothing in the world but matter in motion, and matter in motion cannot move otherwise than in space and time." (V. I. Lenin. *Collected Works*, Vol. 14, p. 175, 1962.)

The mechanical motion of a material body can only be observed and studied in relation to other bodies. If only one body were in motion in space and there were no other bodies, then we would not be able to judge any changes in the position of the given body and, hence, we would not be able to say anything about its motion.

The system of coordinates fixed in the body and with respect to which the motion is considered is termed a reference frame.

The motion of one and the same body relative to different reference frames may be quite different. For example, to an observer on board a ship, an object lying on the deck is at rest, yet that same object is in motion when viewed by an observer on the shore. A stone thrown vertically upwards from the deck of a ship in uniform motion falls back to the same spot and, hence, moves in a straight line relative to the deck. Now to an observer on the shore, the stone moves in a curved line (parabola). If some body is at rest relative to the earth, it is in motion relative to the sun, and so on. There are no bodies in nature absolutely at rest and therefore it is fundamentally impossible to establish any kind of absolutely fixed frame of reference.

Thus the concepts of motion and rest are relative notions and have meaning only when the reference frame is given, with respect to which they are being considered. If a body alters its position relative to a chosen frame of reference, that body is said to be in motion relative to the given frame of reference. If the position of the body relative to a reference frame remains unchanged with time, we say that the body is at rest relative to the given reference frame. *In engineering practice, the basic (or "fixed") reference frame that is ordinarily taken is that which is fixed relative to the earth, and the motion of bodies relative to that frame of reference is taken (by agreement) to be absolute.* One must bear in mind that for the purposes of kinematics which considers the motion of bodies irrespective of forces acting on them, it is inconsequential whether the "fixed" system is in motion or not.

The motion of a body is said to be known if we are able to determine its position relative to the chosen frame of reference at any given instant of time.

Time is a scalar quantity continually changing and is regarded in kinematics problems as the independent variable (argument). All other quantities that vary with time are considered to be functions of time.

Just as the position of a point on a straight line is determined by its distance from some other point, so a *given instant of time t* is determined by the interval of time that elapses from some other instant, which is taken, in the given problem, to be the *initial instant of time*, that is, the starting point from which the time is reckoned. The zero value of the variable t may be taken as the initial instant of time. Then times t_1 and t_2 will be determined by the time intervals t_1 and t_2 respectively that separate them from the initial instant of time, and the *time interval* between times t_1 and t_2 by the difference $t_2 - t_1$.

Generally, different points of a body may perform different motions. Therefore the study of motion ordinarily begins with the so-called kinematics of a point (particle), that is, with the establishment of the kinematic characteristics of motion of a separate material point (particle). Such an approach to the study of kinematics is not an abstraction divorced from practical matters. The motion of every body is made up from the motions of the separate particles of the body. If we are able to determine the motion of a separate particle, we can then find the motion of the whole body. Besides, as we shall see in what follows, it suffices to know the motion of only one of the particles of a body in order to judge the motion of the body as a whole.

In the process of its motion, a particle successively occupies various positions relative to the frame of reference.

The curve described by a particle (point) moving in space is termed the path (or trajectory) of that particle (point). The motion of a point is said to be *rectilinear* if its path is a straight line, and *curvilinear* if its path is a curved line. Depending on the shape of the curve, curvilinear motion can in turn be diverse: circular (when the path of the point is a circle or an arc of a circle), elliptical (when the path of the point is an ellipse), helical (when the path of the point is a helix), and so on.

The shape of the path will of course depend on the choice of reference frame. That becomes evident even from the case of the stone thrown up from the deck of a moving ship. Any classification of motions is of a relative nature and is meaningful only when they are considered relative to one and the same frame of reference.

If a point traverses the same distances in equal and arbitrarily chosen time intervals, then the motion of the point is described as uniform motion, otherwise we have nonuniform (or varying) motion.

The motion of a point is described by features that are established for each of the two classifications given above. Both rectilinear and curvilinear motion of a point may simultaneously be either uniform motion or nonuniform (varying) motion.

Sec. 54. Methods of specifying the motion of a point

1. Natural method

Suppose a point is in motion along some path AB (Fig. 138). On that path, take an arbitrary (fixed) point O and call it the origin of the reference frame for reckoning distances. *The arc length OM measured in linear units is termed the distance of point M from the origin, or the arc coordinate.* We denote this distance by a lower case letter s (to distinguish it from the path traversed by the point, which is denoted by a capital S). For the sake of definiteness we

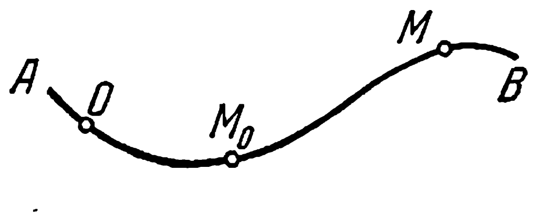


FIG. 138

have to agree on the direction of reckoning distances and regard the distance s as an algebraic quantity: positive, if the distance is reckoned along the path in one direction, from the origin O, and negative, if it is reckoned in the opposite direction.

Generally, the choice of the positive direction is arbitrary.

Note that the distance s of a point from the origin should not be confused with the path length S traversed by the point during a specified time interval.

The path S traversed by a point during some time interval t is readily found when the point is moving in one direction along its path (ordinarily, this direction is taken as the positive direction for reckoning distances). If at the initial time ($t = 0$) the point was at the origin, then, clearly, the path S covered by the point during the time interval t is equal to the distance s from the origin, which is to say $S = s$ here. However, if at time $t = 0$ the point was not at the origin O but at some position M_0 (Fig. 138) determined by the distance $s_0 = \overset{\frown}{OM_0}$ from the origin (this distance is called the *initial distance*), then the path traversed by the point during the time interval t is $S = s - s_0$.

But if the direction of motion of the point along the path varies, its distance s from the origin will in the course of time increase and then decrease, whereas the path S traversed by the point can only increase. Referring to Fig. 138, suppose a certain point moved along its path during a time interval from 0 to t from the origin O to M and back again. Then, at time t, the distance of the point from the origin is $s = 0$, whereas the distance traversed by the point during the time interval t is equal to twice the arc OM : $S = 2\overset{\frown}{OM}$.

At each instant, a point can occupy only one definite position on the path and, hence, its distance s from the origin is a certain unique function of the time t. The relationship between the variables s and t may be expressed in common mathematical symbols by the equation

$$s = f\ (t). \tag{57}$$

This equation which expresses the functional relationship between the distance s of a point from the origin and the time t of its motion is called the *equation of motion of the point along the given path (trajectory)* or *the law of motion.*

If the law of motion of a point along a given path has been established, that is, if s is a known function of t, then at any time we can determine the distance of the point from the origin and thus establish its position on the path.

The path of a point may be specified in different ways: either analytically, that is, in the form of an equation of a curve, or geometrically. The function $s = f(t)$, which is the law of motion of the point along a path, can also be specified either analytically or in the form of a graph. The graph of the function $s = f(t)$ is called the *curve of distances or the graph of motion.*

This graph, which gives a clear-cut picture of the nature of the motion of the point, is easy to construct if the function $s = f(t)$ is known. By specifying the time numerically at various instants, we can substitute the values into the expression of the given function and compute the appropriate values of the distance s. Then (see Fig. 139), we take two mutually perpendicular axes and lay off the values of the argument t along the axis of abscissas (which is then called the *time axis*) and corresponding values* of the distances s on the axis of ordinates (which is then called the *distance axis*). Joining the points constructed by means of coordinates t and s, we obtain a curve which is the graph of the given function $s = f(t)$.

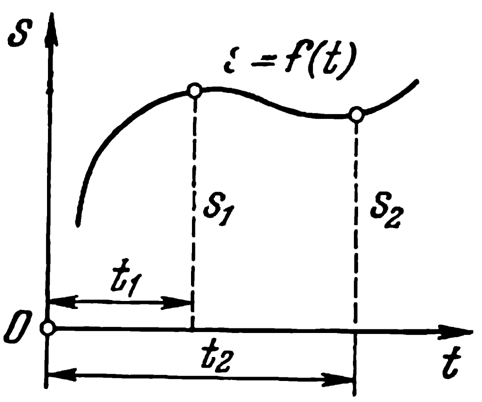


FIG. 139

The importance of the graphical construction of the relationship $s = f(t)$ lies in the fact that it enables one to obtain the approximate equation of motion of a point along a given path even when the values of the distances s are known only for separate times t and the analytical relationship between s and t is not known. Sometimes the curves of the distances are drawn automatically by means of recording devices. If we have the graph of motion, it is always possible to find the distance s of the point from the origin and to determine its position on the path. This path must of course be known, just as in the analytical specification of a function. Note that the distance curve (graph of motion) must not be identified in any way with the trajectory of motion of the point. For example, for uniform motion of point M along some curve as depicted in Fig. 138, the path of the point is the given curve AB, while the graph of motion [the graph of the function $s = f(t)$] is a straight line (since the increment in the distance s of the point M from the origin O is directly proportional to the increment in the time of motion of the point).*

---

* The scales for t and s may be chosen arbitrarily and independently.

This method of determining the position of a point is called the natural method. To summarize: *under the natural method of specifying the motion of a point, three things must be known:* (a) the *path of the point* in a chosen reference frame, (b) the *origin and the positive direction of reckoning*, (c) the *law of motion of the point along the given path* in the form of an equation, $s = f(t)$, or a graph.

2. *Coordinate method*

The coordinate method of specifying the motion of a point is based on the fact that the position of the point relative to some reference frame can always be given with the aid of a definite set of numbers called coordinates of the point.

A variety of systems of coordinates are in use. For example, the position of a point on the earth's surface is customarily determined with the aid of so-called geographical coordinates: latitude and longitude.

Most widely used is the rectangular coordinate system and so we will describe the method of specifying the motion of a point in that system. Take three mutually perpendicular coordinate axes (Fig. 140). The position of point M relative to the given system of coordinates is quite definitely described by three coordinates of the point: x, y and z. As M

---

* Figure 139 depicts the graph of nonuniform motion of a point. The ordinate s of the graph increases and decreases. Consequently, the point M first recedes from the origin and then approaches it.

moves, its coordinates vary in time, that is, they are certain functions of the argument t:

$$\left.\begin{aligned} x &= f_1(t), \\ y &= f_2(t), \\ z &= f_3(t). \end{aligned}\right\} \qquad (58)$$

If we know how the coordinates of a moving point vary with time, that is if we know equations (58), we will always

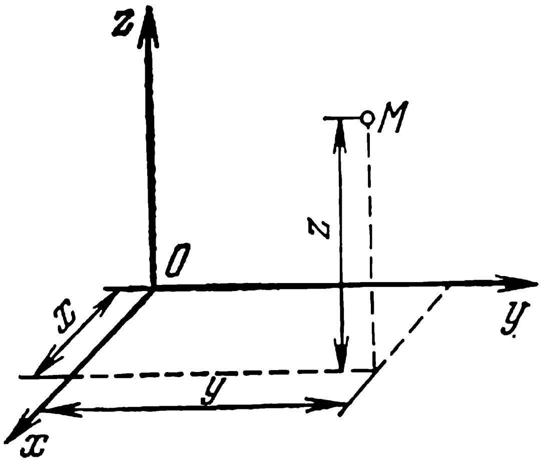


FIG. 140

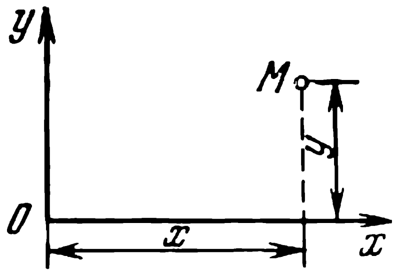


FIG. 141

be able to determine the coordinates of the point for any instant of time and, hence, determine the position of the point relative to the given coordinate system.

The equations (58), which express functional relationships between the rectangular coordinates (x, y, z) of a point and the time t of its motion, are called the *equations of motion of the point in rectangular coordinates.*

If during its motion, the point remains in one plane, then, choosing a system of two mutually perpendicular axes x and y in that plane, we are able to determine any position of the point in the given plane by means of only two coordinates. As point M moves in a single plane (Fig. 141), its coordinates x and y, which fully define the position of the point, will vary with time; that is, they are certain functions of the argument t. Consequently, the plane motion of a point is determined by two equations of motion in rectangular coordinates:

$$\left.\begin{aligned} x &= f_1(t), \\ y &= f_2(t). \end{aligned}\right\} \qquad (59)$$

If a point performs rectilinear motion, then we can take the straight line in which it moves as one of the coordinate axes, say the x-axis. The position of the point M on that axis is fully determined by a single coordinate $OM = x$ (Fig. 142) and, hence, the rectilinear motion of a point is given by a single equation:

$$x = f(t). \tag{60}$$

If the point is in motion along a straight line in one direction and at the initial time was at the coordinate origin O, then the absolute value of the coordinate x will at the same time be the length of the path traversed, S.

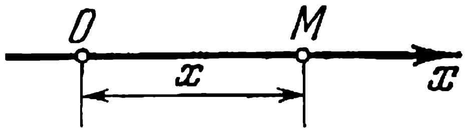


FIG. 142

Equations (58) and (59) of the motion of a point in rectangular coordinates determine the position of a moving point at any instant of time and thus determine its path. By eliminating the time t from the given equations of motion of the point, we obtain the equation of its path, that is, an equation that connects the running coordinates of the moving point (in the case of spatial motion, we obtain the equations of two surfaces whose intersection determines the path).

The path of a point may also be found graphically; this is done by constructing, on the basis of the given equations of motion of the point, a series of successive positions and then joining them with a continuous curve.

In many practical problems the motion of a point is determined by the geometric conditions that follow from the given design of a mechanism. These conditions are used (analytically and graphically) to find the path and the equations of motion of the point of interest.

Establishing the quantities that determine, at any time, the position of a moving point relative to a chosen reference frame does not however exhaust the problem of a kinematic investigation of the motion of a point. The most important characteristics of any motion of a point are its velocity and acceleration. We will devise methods for determining them in the following chapters. For the present let us merely note that they may be found by means of the natural method and also using the coordinate method of specifying the motion of a point. Both methods give a complete description of the motion and permit finding all its characteristics.

Problem 57. A crank *OA* (Fig. 143) is in uniform rotation about a fixed axis *O* and sets in motion a slide block *B* by means of a connecting rod *AB* hinged to the crank and the slide block. The angle φ of rotation of the crank varies in time via the law $\varphi = kt$. Use rectangular coordinates and determine the equations of motion of the midpoint *M* of the connecting rod and find the path of that point. Length of crank $OA = AB = l$. (The fixed guide path of the slide block is located behind the crank gear.)

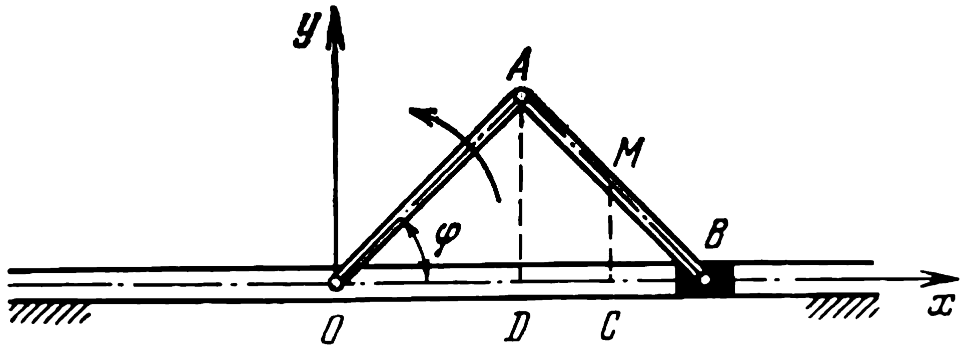


FIG. 143

Solution. Draw the rectangular coordinate axes x and y as shown in Fig. 143. $\triangle OAB$ is an isosceles triangle and so $\angle ABO = \angle AOB = \varphi$.

The coordinates of point *M* in the given coordinate system are

$$x = OC = OD + DC = OA \times \cos\varphi + AM \times \cos\varphi = \frac{3}{2}\, OA \times \cos\varphi$$

and

$$y = MC = \frac{1}{2}\, AB \times \sin\varphi.$$

Substituting into these equations the values $OA = AB = l$ and $\varphi = kt$, we obtain the desired equations of motion of the midpoint *M* of the connecting rod:

$$x = 1.5l\cos kt, \qquad y = 0.5l\sin kt.$$

To find the path of point *M* we eliminate the time t from its equations of motion. To do this, take the given equations and find the values of $\cos kt$ and $\sin kt$, square them and add. Since the sum of the squares of the sine and the cosine of one and the same argument is equal to unity, we have

$$\cos^2 kt + \sin^2 kt = \frac{x^2}{(1.5l)^2} + \frac{y^2}{(0.5l)^2} = 1,$$

Thus, the path of the midpoint M of the connecting rod is a curve given by the equation $\frac{x^2}{(1.5l)^2}+\frac{x^2}{(0.5l)^2}=1$. This curve is clearly an ellipse with centre at the fixed point O and with semiaxes $a = 1.5l$ and $b = 0.5l$.

Problem 58. Referring to Fig. 144, a particle M of soil coming off the belt of a horizontal conveyer located at a height of $h = 122.5$ cm above the earth's surface is in motion in accord with the equations $x = 2t$, $y = 4.9t^2$. (In these equations, t is expressed in seconds, and x and y in metres.) The coordinate origin with respect to which the motion of point M is given in the equations is taken at point O, the fly-off point of the particle; the x-axis is horizontal and the y-axis is directed vertically upwards. Find the equation of the path of the particle of soil M, its flight distance l to impact, and the time of flight T.

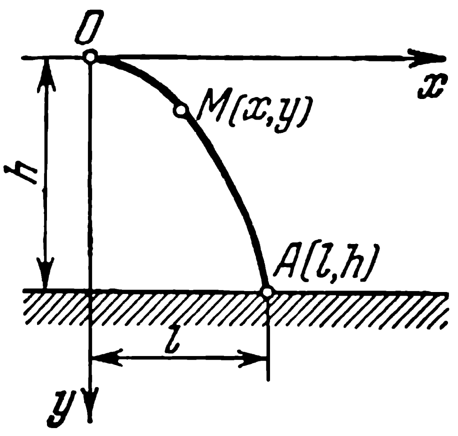


FIG. 144

Solution. To find the path of point M, eliminate t from the equations of motion. From the first equation we find $t = x/2$. Substituting this value of t into the second equation, we obtain the equation of the path: $y = 4.9t^2 = 4.9\,\frac{x^2}{4} = 1.225x^2$ or $y = 1.225x^2$. The desired path is clearly a parabola symmetric about the y-axis and with vertex at O. This equation must of course be satisfied by the coordinates of any point of the path of a falling particle of soil. The coordinates of point A, the point of intersection of the given path and the earth's surface, are: h—the height of point O, which is the fly-off point of the particle, and l—its range (distance of flight to impact). Substituting into the path equation $y = h = 122.5$ cm $= 1.225$ m and $x = l$, we get $1.225 = 1.225l^2$, whence $l = 1$ m. Then, substituting $x = l = 1$ m and $t = T$ into the first of the given equations of motion of the point (that is, into the equation $x = 2t$), we also find the time of flight of the particle of soil: $1 = 2T$, whence $T = 0.5$ s.

CHAPTER XI

THE VELOCITY OF A POINT

Sec. 55. The concept of the velocity of a point

The velocity of a point is a vector* which at every given time determines the speed and direction of the moving point.

The simplest case is that of determining the velocity of a point in uniform rectilinear motion.

The ratio (which is a constant for the given uniform motion) *of the path S traversed by the point during a certain time interval to the value t of that interval is the magnitude v of the velocity of the point in uniform motion*:

$$v = \frac{S}{t}. \tag{61}$$

In the case of rectilinear motion, for the direction of the velocity vector $\mathbf{v}$ of the point we take the direction along the path, that is, the *velocity vector is directed along the rectilinear path of the particle from the given position of the point in the direction of its motion.*

From formula (61) it follows that the velocity has the following dimensions**:

$$[v] = \frac{[\text{length}]}{[\text{time}]}.$$

With each choice of units of length and time there is associated a unit of velocity. Velocity may be expressed in m/s, cm/s, m/min, km/h and so forth.

---

* Confirmation of the fact that the velocity of a particle is a vector quantity is the possibility (which is proved in Sec. 70) of the geometrical composition of velocities.

** The dimensions of a quantity are given as an expression of its units of measurement in terms of the basic measuring units. The dimensions of a quantity are usually given in square brackets.

Let us now extend the concept of velocity to the case of any curvilinear and nonuniform motion.

Suppose a certain point is in motion along a curvilinear path AB (Fig. 145). At successive instants of time t and $t+\Delta t$* it occupies positions M and M' respectively. Join the points M and M' by a vector.

The vector $\overrightarrow{MM'}$ that joins the position of the point at the start of its time interval Δt with the position at the end of that interval is termed the displacement vector of the point for that time interval.

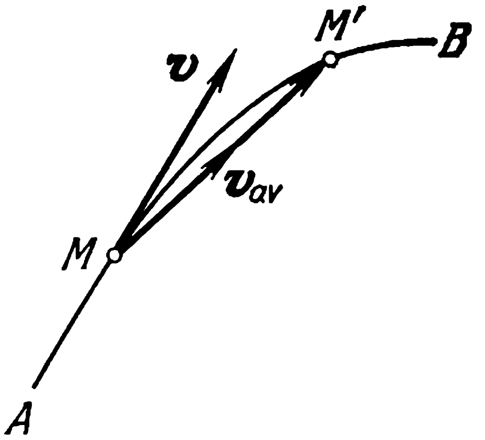


FIG. 145

If we imagine the point M to be moving uniformly not along the arc $\overset{\frown}{MM'}$ (as it actually does) but along the chord MM', then the velocity of that imaginary motion is termed the *average velocity vector (for the time interval Δt) of the point*:

$$v_{\mathrm{av}}=\frac{\overrightarrow{MM'}}{\Delta t}. \tag{62}$$

The displacement $\overrightarrow{MM'}$ of the point is a vector. But the division of a vector by a positive quantity Δt yields a vector of the same direction with a magnitude equal to the magnitude of the given vector divided by Δt.

Thus, *the average velocity vector* $\mathbf{v}_{\mathrm{av}}$ *of a point over a time interval Δt has the direction of the displacement vector $\overrightarrow{MM'}$ and is equal in magnitude to the quotient obtained by dividing the magnitude MM' of the displacement vector of the point by that time interval.*

It is obvious that the average velocity of a point only permits judging the direction of a finite displacement of the point over the time interval under consideration and the average speed of that displacement, but it does not give any idea of the actual velocity of motion of the point at each instant of time.

---

* The Greek letter Δ (delta) is not a multiplier of t but only a symbol indicating a small increment in the time t.

If the time interval Δt is changed, so also is the displacement of the point $\overrightarrow{MM'}$. Generally speaking, the average velocity of the point is a variable quantity that depends on the chosen magnitude of the time interval Δt. But the smaller we take the interval Δt, the greater the accuracy attained in replacing the small path length (arc $\overset{\frown}{MM'}$) by the chord MM', and the more exact the average velocity of the point over that time interval in characterizing the speed and direction of motion of the point at time t, that is, at the instant that the point occupies position M on the trajectory.

As the time interval Δt is reduced, so the direction and magnitude of the displacement vector $\overrightarrow{MM'}$ of the point changes, but the ratio $\overrightarrow{MM'}/\Delta t$ tends to a certain definite limit.

From this it follows that the *velocity vector* v *of a point is equal, at a given time, to the limit of its average velocity over the time interval starting at the given time, when the magnitude of that time interval tends to zero*:

$$\mathbf{v} = \lim_{\Delta t \to 0} \mathbf{v}_{\text{av}} = \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t}. \tag{63}$$

Since the vector $\mathbf{v}_{\text{av}}$ is directed along the displacement vector $\overrightarrow{MM'}$ of the point and, in the limit, as Δt decreases without bound, the secant line $\overrightarrow{MM'}$ coincides with the tangent at the point M, it follows that the *velocity vector* v *of the point is at very instant of time directed along the tangent to the trajectory at the given point and in the direction of motion of the point* (Fig. 145).

Methods for determining the magnitude of the velocity of a point at a given time depend on the manner in which the motion of the point is specified and will be established in the following sections.

Sec. 56. Determining the velocity of a point in the natural method of specifying motion

In the natural method of specifying the motion of a point, we have to know the path AB (Fig. 146) and the law of motion of the point along that path, $s = f(t)$, where s is the arc coordinate, that is, the distance of the point from the origin. The positive direction of reckoning distance is indicated in Fig. 146 by a plus sign.

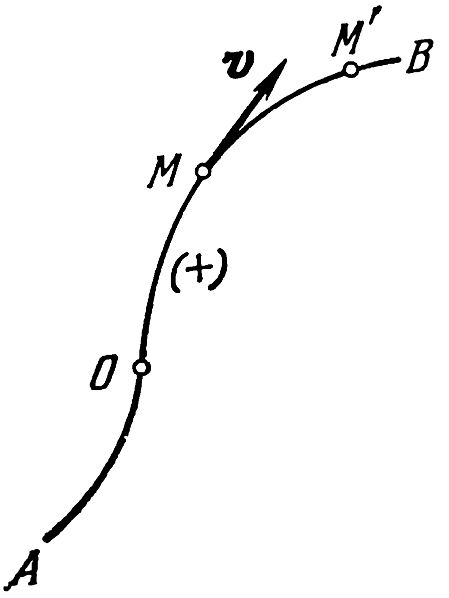


FIG. 146

Since the path of the point is known in this method of specifying the motion of a point, the direction of the velocity vector **v** is also known. As mentioned earlier, it is always in the direction of motion of the point along the tangent to the path at the given point (Fig. 146).

It remains to determine the magnitude (absolute value) of the velocity of the point.

Suppose that at time t the point was at position M on the path, and at the time $t + \Delta t$ (Δt is a small time interval) at position M'. At time t, the distance of the moving point from the origin is the arc length $OM = s = f(t)$, and at time $t + \Delta t$, it is the arc length $OM' = s' = f(t + \Delta t)$.

Denoting the length of arc $\overset{\frown}{MM'}$ of the path covered by the point during the time interval Δt by Δs, we have $\Delta s = s' - s = f(t + \Delta t) - f(t)$, which is to say Δs is equal to the increment in the given time function over the time interval Δt.

The numerical value of the velocity of the point at the given time t is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

As we know from mathematics, the limit of the ratio of the increment in a function to the corresponding increment in the argument, as the latter tends to zero, is termed the derivative of the given function. Using the usual notations for a derivative, we can write

$$v = \frac{ds}{dt} = f'(t).$$

If, as we originally assumed, the point is in motion along the path in the direction of increasing distances s (in the direction of reckoning positive distances), the increment Δs in the distance of the point from the origin is positive. In that case, the derivative ds/dt is positive and does indeed yield the magnitude v of the velocity.

Now, if the point is in motion in the direction of diminishing distances s (in the direction of a negative reckoning of the distances s), then the increment Δs is negative. In that case, the derivative ds/dt is negative. Generally, in both cases the derivative ds/dt yields the so-called *algebraic value of the velocity*, which not only determines the magnitude (absolute value) of the velocity but also the direction of motion along the path (trajectory).

For the sake of simplicity, from now on we will not distinguish notationally between the magnitude (absolute value) of the vector and its algebraic value. Merely bear in mind that the magnitude of any vector is always a positive quantity and is equal to the absolute value of the algebraic magnitude of the vector. If they have to be distinguished, we will denote the magnitude of the vector by enclosing it between vertical bars.

If the time interval Δt that follows time t is so small that the direction of motion of the point during that interval may be regarded as unchanged, then the absolute value of the increment $|\Delta s|$ in the distance of the point from the origin O is equal to the increment ΔS in the path traversed by the point during the same time interval Δt. Also, obviously, the path S covered by the point is some function of the time. The increment ΔS in the path is of course always positive. Then

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt}.$$

Combining this equation with the preceding one, we have

$$v=\frac{ds}{dt}=\frac{dS}{dt}. \tag{64}$$

The numerical value of the velocity at a given time is equal to the derivative, with respect to time, of the arc coordinate s of the point or of its path length S.

To compute the magnitude of the velocity of a point from the given equation of its motion $s = f(t)$ at some time $t = T$, take the derivative of the function s with respect to the time t and then substitute into the equation of the derivative thus found the particular value $t = T$ of the argument t that corresponds to the time at which the velocity is being determined. The absolute value of the result obtained is the magnitude of desired velocity of the point at time T.

Problem 59. A point is in motion along a specified curvilinear path in accordance with the law $s = 5 \sin \pi t$ (t is in seconds and s in centimetres). Find the magnitude and the direction of the velocity at time $t = 3$ s.

Solution. For $t = 0$, $s = 5 \sin (\pi \times 0) = 0$. At the initial time, the point is at the origin of the reference frame. At $t = 3$ s, $s = 5 \sin 3\pi = 5 \sin \pi = 0$; thus the point returns to the origin. The numerical value of the velocity of the point $v = \frac{ds}{dt} = 5\pi \cos \pi t$ cm/s. Substituting the derivative of the value $t = 3$ s into this expression, we get

$$v = 5\pi \cos 3\pi = 5\pi \cos \pi = 5\pi \times (-1) \approx -15.7 \text{ cm/s}.$$

The negative value of the derivative ds/dt at $t = 3$ s shows that at that time the point is moving along its path in the opposite direction to that taken as positive for reckoning the distance s in the reference frame. We thus have: the desired velocity of the point at time $t = 3$ s is 15.7 cm/s and is directed along the tangent to its path in the direction opposite to the positive side of the reference frame.

Sec. *57. Finding the velocity of a point from the equations of motion in rectangular coordinates

Suppose a point is in motion in the plane xOy in accordance with the specified equations of motion: $x = f_1(t)$ and $y = f_2(t)$. Referring to Fig. 147, note the positions M and

M' of the moving point at times t and $t+\Delta t$. The coordinates of these points are x, y and $x+\Delta x$, $y+\Delta y$. Joining the points M and M', we obtain the displacement vector $\overrightarrow{MM'}$ of the point over the time interval Δt. The vector of the average velocity of the point during the time interval [formula (62)] is

$$\mathbf{v}_{\text{av}}=\frac{\overrightarrow{MM'}}{\Delta t}.$$

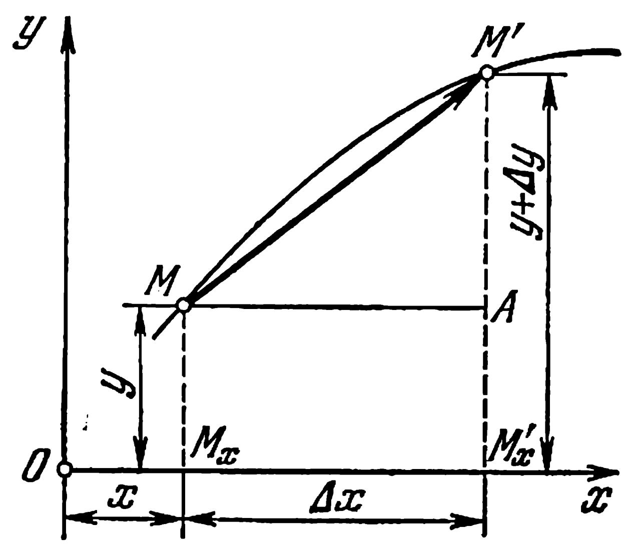


FIG. 147

From point M draw the straight line MA parallel to the x-axis. As is evident from Fig. 147, the projection of the vector $\overrightarrow{MM'}$ on x-axis is

$$(\overrightarrow{MM'})_x=\Delta x.$$

Consequently, the projection of the average velocity of the point on that axis is

$$v_{\text{av }x}=\frac{(\overrightarrow{MM'})_x}{\Delta t}=\frac{\Delta x}{\Delta t}.$$

Passing to the limit as $\Delta t\to 0$ and denoting the projection of the true velocity of the point on the x-axis by v_x, we obtain

$$v_x=\lim_{\Delta t\to 0} v_{\text{av }x}=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}.$$

But the coordinate x is a function of the time t; Δx is an increment of this function that corresponds to the increment Δt in the argument t, and $\lim\limits_{\Delta t\to 0}\frac{\Delta x}{\Delta t}$ is the derivative of this function. Hence

$$v_x=\frac{dx}{dt}=f_1'(t). \tag{65a}$$

Reasoning in similar fashion, we find the projection of the velocity of the point on the y-axis:

$$v_y = \frac{dy}{dt} = f_2'(t). \tag{65b}$$

The projections of the velocity of the point on the fixed coordinate axes are equal to the first derivatives of the appropriate coordinates of the moving point with respect to time.

As the point moves during time Δt along an arc of the path from M to M', the projection of this point on the x-axis moves from M_x to M'_x covering the distance $M_xM'_x = \Delta x$. Clearly, the distance Δx and the derivative $v_x = dx/dt$ are positive if the projection of the point on the axis moves in the positive direction of the axis (see Fig. 147) and it is negative if the projection of the point on the axis moves in the opposite direction.

Thus, the *projection of the velocity of the particle on a coordinate axis determines the magnitude and the direction of the velocity of the projection of the point on that axis.*

A knowledge of the projections of the velocity vector of a point on two fixed mutually perpendicular axes in the plane of which the given vector lies permits finding (as is the case for any vector, see Sec. 11) the magnitude of the velocity vector and its direction:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,. \tag{66}$$

The direction of the velocity vector is given by its direction cosines:

$$\cos(\widehat{\mathbf{v}, x}) = \frac{v_x}{v}, \quad \cos(\widehat{\mathbf{v}, y}) = \frac{v_y}{v}. \tag{67}$$

If a point is in motion in space and not in a plane, then a third equation, $z = f_3(t)$, is needed to determine its motion.

As before, we can find the projection of the velocity of the point on a third coordinate axis, $v_z = dz/dt$, and then, on the basis of the three projections of the velocity vector $\mathbf{v}$ on the three mutually perpendicular axes, we can find its magnitude and direction cosines (just as, in Sec. 39, we found the magnitude and direction of the force vector on the basis of its three projections on the coordinate axes). From now on, however, we will confine ourselves to a consideration of the most frequently encountered type of motion, the motion of a point in a plane.

Problem 60. The motion of a shell is given by the equations $x = 400\sqrt{2}t$ and $y = -5t^2 + 400\sqrt{2}t$ (x and y are in metres, t is in seconds). Find (1) the equation of the trajectory, (2) the height h and the range l, (3) the velocity v_1 at the highest point in the flight path and the velocity v_2 at the time of impact (when the shell hits the earth).

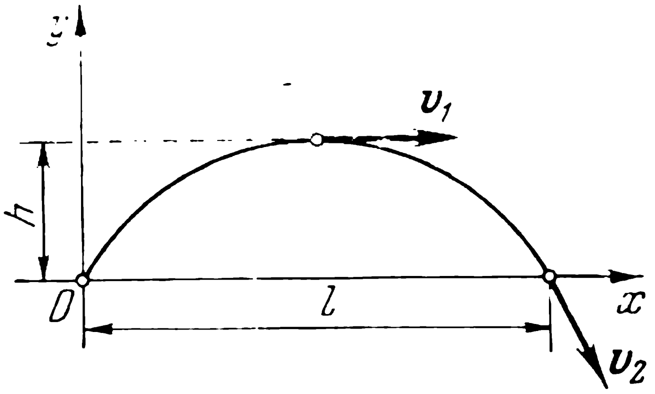


FIG. 148

Solution. (1) To find the flight path of the shell, we eliminate the time from its equations of motion:

$$t = \frac{x}{400\sqrt{2}}, \quad t^2 = \frac{x^2}{320\,000}, \quad y = -5t^2 + 400\sqrt{2}t$$

$$= -5\frac{x^2}{320\,000} + x.$$

The flight path of the shell is a parabola (Fig. 148) given by the equation

$$y = -\frac{x^2}{64\,000} + x.$$

(2) The flight path of the shell intersects the x-axis at two points at which the ordinate y is zero. Substituting this value of y into the flight-path equation, we obtain the abscissas of the points of intersection:

$$0 = x - \frac{x^2}{64\,000},$$

whence

$$x_1 = 0, \quad x_2 = 64\,000 \text{ m}.$$

The range is clearly $l = x_2 = 64$ km. To determine the maximum height $h = y_{\max}$ of the flight path we could take advantage of the general procedure for finding the maximum of a function, $y = f(x)$, but it is easier to find the desired height from the symmetry property of the curve by substituting into the flight-path equation

$$x = \frac{l}{2} = 32\,000 \text{ m} \quad \text{and} \quad y = h;$$

$$h = 32\,000 - \frac{32\,000^2}{64\,000} = 16\,000 \text{ m} = 16 \text{ km}.$$

(3) The projections of the velocity of the shell on the coordinate axes are found from formulas (65):

$$v_x = \frac{dx}{dt} = (400\sqrt{2}\,t)' = 400\sqrt{2} \text{ m/s},$$

$$v_y = \frac{dy}{dt} = (-5t^2 + 400\sqrt{2}\,t)' = -10t + 400\sqrt{2} \text{ m/s}.$$

As we see, the projection of the velocity of the shell on the x-axis is constant, whereas the projection of the velocity on the y-axis depends on the in-flight time of the shell.

At the highest point of the flight path the velocity vector $\mathbf{v}_1$ of the shell, which is tangential to the path at that point, is parallel to the x-axis. Hence

$$v_{1y} = 0, \quad v_1 = v_{1x} = 400\sqrt{2} \text{ m/s}.$$

To find the velocity $\mathbf{v}_2$ of the shell at time t_2 (time of impact), one has to find the time of flight of the shell. To do this, substitute into the equation of motion, $x = 400\sqrt{2}t$, the values $x = l = 64\,000$ m and $t = t_2$. This yields $64\,000 = 400\sqrt{2}t_2$ and

$$t_2 = \frac{64\,000}{400\sqrt{2}} = 80\sqrt{2} \text{ s}.$$

Substituting the value of t_2 thus found (time of flight of the shell) into the expressions for the projections of the velocity of the shell, we obtain

$$v_{2x} = 400\sqrt{2} \text{ m/s},$$

$$v_{2y} = (-10t_2 + 400\sqrt{2}) = -10 \times 80\sqrt{2} + 400\sqrt{2} = -400\sqrt{2} \text{ m/s}.$$

From this we get the magnitude of the desired velocity:

$$v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{(400\sqrt{2})^2 + (-400\sqrt{2})^2} = 800 \text{ m/s}.$$

The direction of the vector of this velocity is found from the equations

$$\cos\widehat{(\mathbf{v}_2, x)} = \frac{v_{2x}}{v_2} = \frac{400\sqrt{2}}{800} = \frac{\sqrt{2}}{2},$$

$$\cos\widehat{(\mathbf{v}_2, y)} = \frac{v_{2y}}{v_2} = -\frac{400\sqrt{2}}{800} = -\frac{\sqrt{2}}{2},$$

whence we obtain the angles

$$\widehat{(\mathbf{v}_2, x)} = 45^\circ, \qquad \widehat{(\mathbf{v}_2, y)} = 135^\circ.$$

CHAPTER XII

THE ACCELERATION OF A POINT

Sec. 58. The concept of acceleration of a point

The motion of a point with a constant (in magnitude and direction) velocity (uniform rectilinear motion) rarely occurs in practical situations: only when the impressed forces mutually balance. In the majority of cases, the velocity of a point varies during its motion. This variation may be confined to the magnitude (nonuniform rectilinear motion) or to the direction (uniform curvilinear motion), or it may involve both magnitude and direction (in the general case of nonuniform curvilinear motion). In order to establish, in dynamics, the relationship between the variation in the motion of a body and the cause of such variation by forces acting on the body, it is necessary to find some way of describing the change of motion and to establish a measure of such change.

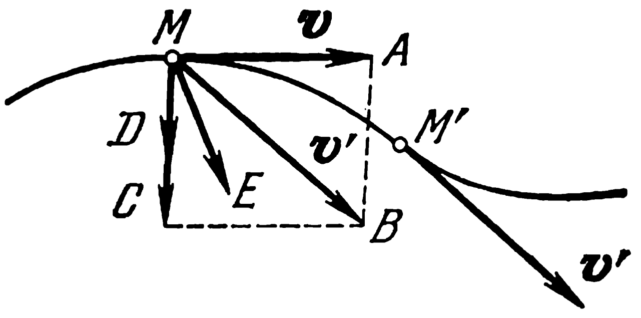


FIG. 149

The quantity characterizing the rate of change of the velocity vector in magnitude and direction is called acceleration.

Referring to Fig. 149, suppose a point moving along a curved path occupies position M at time t and position M' at time $t + \Delta t$. The velocity vectors $\mathbf{v}$ and $\mathbf{v}'$ of the point at M and M' will be directed along the appropriate tangents.

Let us determine the change in velocity vector of a point* during a time interval Δt, that is, during the time the point moves from M to M'. To do this, carry the origin of the vector $\mathbf{v}'$ to M, join the end points A and B of the vectors $\mathbf{v}$ and $\mathbf{v}'$ and complete the triangle MAB to make a parallelogram $MABC$. The vector $\overrightarrow{MB} = \mathbf{v}'$, which is a diagonal of the parallelogram, is the geometric sum of the vectors $\overrightarrow{MA} = \mathbf{v}$ and $\overrightarrow{MC}$, and for that reason the vector $\overrightarrow{MC}$ is the geometric difference* between the vectors $\mathbf{v}'$ and $\mathbf{v}$:

---

* In curvilinear motion of a point, the velocity vectors $\mathbf{v}$ and $\mathbf{v}'$ are always distinguishable. In uniform curvilinear motion, the vectors $\mathbf{v}$ and $\mathbf{v}'$ have the same magnitudes but differ in direction. In the nonuniform curvilinear motion of a point, the vectors $\mathbf{v}$ and $\mathbf{v}'$ differ in magnitude and in direction.

$$\mathbf{v}' = \mathbf{v} + \overrightarrow{MC}, \quad \text{whence} \quad \overrightarrow{MC} = \mathbf{v}' - \mathbf{v} = \Delta\mathbf{v}.$$

If we geometrically add to the velocity vector $\mathbf{v}$ of a point at time t the increment $\Delta\mathbf{v}$ of that vector over the time interval Δt, we obtain the vector $\mathbf{v}'$ of the point at time $t + \Delta t$. *The vector* $\Delta\mathbf{v}$, *which is the geometric difference between the velocity vectors at the end and the beginning of the given time interval, is termed the increment (vectorial or geometric) in velocity of the point over the appropriate time interval.*

The vector $\Delta\mathbf{v}$ completely determines the change in velocity of a moving point both in magnitude and in direction during the time Δt, therefore the ratio of this vector to the given time interval is the average acceleration of the point during that time interval.

We denote the vector of average acceleration of a point by the symbol $\mathbf{a}_{\text{av}}$ and we have

$$\mathbf{a}_{\text{av}} = \frac{\Delta\mathbf{v}}{\Delta t}. \tag{68}$$

---

* The geometric difference between two vectors is a third vector which, when geometrically combined with the subtrahend, yields the minuend (vector).

Note that the term minuend is used here merely as an analogy with arithmetic because the concepts of "greater" and "less" are not applicable to vectors and can only be applied to their magnitudes. But the magnitude of a geometric difference of vectors can be greater than the magnitude of the "minuend" vector.

By dividing the vector $\Delta\mathbf{v} = \overrightarrow{MC}$ by the scalar quantity Δt, we obtain the vector $\overrightarrow{MD}$ of the same direction as the vector $\Delta\mathbf{v}$ but with magnitude $|\Delta\mathbf{v}/\Delta t|$. If $\Delta t < 1$, the magnitude of the vector $\overrightarrow{MD}$ is greater than that of the vector $\overrightarrow{MC}$. But the magnitudes of the vectors $\Delta\mathbf{v} = \overrightarrow{MC}$ and $\mathbf{a}_{\mathrm{av}} = \overrightarrow{MD}$ have different dimensions and independent scales.

Thus, *the average acceleration of a point over a given time interval is a vector equal to the ratio of the vector of the increment in velocity of the point over the appropriate time interval to the magnitude of that interval.*

If we know the average acceleration $\mathbf{a}_{\mathrm{av}}$ of a point during a certain finite time interval Δt, it is always easy to find the increment $\Delta\mathbf{v}$ in the velocity vector of the point, which increment characterizes the change in magnitude and direction of velocity over that time interval. Indeed, if the vector $\mathbf{a}_{\mathrm{av}} = \Delta\mathbf{v}/\Delta t = \overrightarrow{MD}$ is known, then by multiplying it by the corresponding time interval Δt, we obtain the vector $\Delta\mathbf{v} = \overrightarrow{MC}$.

However, the average acceleration of a point permits examining only the final change in the velocity vector of the point over the time interval under consideration and does not give us any idea about the actual variation of the magnitude and direction of velocity of the point at each instant of time. To each new time interval Δt there corresponds a specific position M' of the point on the trajectory and, generally, the direction and the magnitude of the velocity vector $\mathbf{v}'$ of the point. To any change in the vector $\mathbf{v}'$ there is also associated a change in the vector $\Delta\mathbf{v}$ (the increment in velocity over the time interval in question).

Thus, the average acceleration $\mathbf{a}_{\mathrm{av}}$ of a point during some time interval depends on the magnitude of the interval. If we choose a time interval Δt and then reduce it, the average acceleration of the point $\mathbf{a}_{\mathrm{av}} = \Delta\mathbf{v}/\Delta t$ will vary both in magnitude and in direction. However, as Δt approaches zero, the vector of the average acceleration of the point tends to a certain definite limit, which is termed the true acceleration of the point at the given instant of time or, more often, we simply call it the acceleration of the point.

Denoting the acceleration of a point at a given time by the symbol **a**, we have

$$\mathbf{a} = \lim_{\Delta t \to 0} \mathbf{a}_{\mathrm{av}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}. \qquad (69)$$

The acceleration **a** *of a point at a given instant of time t is equal to the limit of its average acceleration during the time interval Δt beginning at the given time t, when the magnitude of the time interval tends to zero.*

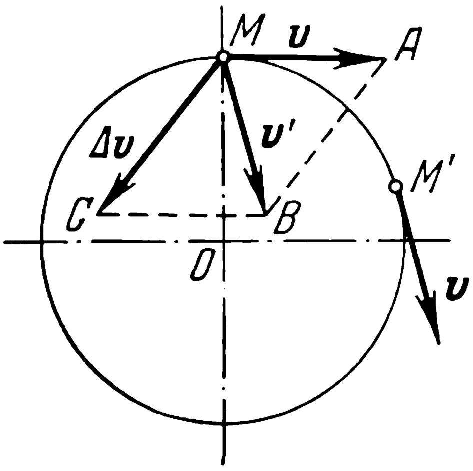


FIG. 150

The acceleration vector **a** of a point at a given time is the limiting value of the vector of average acceleration $\mathbf{a}_{\mathrm{av}} = \Delta \mathbf{v}/\Delta t$ as $\Delta t \to 0$ and its direction, generally, does not coincide* with the direction of the vectors $\mathbf{a}_{\mathrm{av}}$ and $\Delta \mathbf{v}$.

Note that in all cases (with the exception of rectilinear motion) the magnitude of the vector $\Delta \mathbf{v}$ (the velocity-increment vector) of the point is not equal to the increment in the magnitude of velocity Δv. For instance, in the uniform motion of a point along the circumference of a circle the velocity of the point is constant in magnitude but is all the time varying in direction (Fig. 150). In the given case, the magnitudes v and v' of velocity of the point at times t and $t + \Delta t$ are equal: $v = v'$ and $\Delta v = v' - v = 0$. But the vectors **v** and **v**′ of these velocities are distinct, and so $\Delta \mathbf{v} = \mathbf{v}' - \mathbf{v} = \overrightarrow{MB} - \overrightarrow{MA} = \overrightarrow{MC} \neq 0$. Obviously the acceleration of the point, $\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$, is not equal to zero, even though the velocity of the point does not change in magnitude.

---

* In Fig. 149 the vector **a** is conditionally depicted by the vector $\overrightarrow{ME}$.

From this it follows that in the general case the magnitude a of the acceleration $\mathbf{a} = \lim\limits_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$ of the point is not equal to $\lim\limits_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$, that is, it is not equal to the derivative dv/dt of the numerical value of the velocity with respect to time.* We must therefore bear in mind that neither the magnitude nor the direction of the acceleration $\mathbf{a}$ of a point can be obtained directly from formula (69). Ways of obtaining them will be set up in the sections that follow. For the present we only add a few words about the dimensions of acceleration:

$$[a] = \frac{[\text{velocity}]}{[\text{time}]} = \frac{[\text{length}]}{[\text{time}]} : [\text{time}] = \frac{[\text{length}]}{[\text{time}]^2}.$$

With each choice of the units of length and time is associated a unit of acceleration. Acceleration can be expressed in m/s^2, cm/s^2, m/min^2 and so forth.

Sec. 59. Determining the acceleration of a point whose motion is specified by the natural method. Tangential and normal accelerations

If the motion of a point is specified by the natural method, the acceleration vector $\mathbf{a}$ of the point is ordinarily resolved into components directed along the tangent and along the normal to the trajectory of the point. This resolution is convenient for another reason as well. The given components play distinct roles, as we shall see, in the changing motion of the point.

For the sake of simplicity, we will first consider a case where the point is in motion along the arc of a circle. Note the positions M and M' (Fig. 151) of the given point at times t and $t + \Delta t$. Denote by $\mathbf{v}$ and $\mathbf{v}'$ the velocities of the point that correspond to its positions. Carry the origin of the vector $\mathbf{v}'$ from M' to M, join the termini A and B

---

* $\lim\limits_{\Delta t \longrightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t}$ is called the derivative of the vector function. Vector functions are explained in courses of vector analysis.

of vectors **v** and **v**′ and complete the triangle *MAB* to a parallelogram *MABC*.

In accordance with what we said earlier (Sec. 58), the vector $\overrightarrow{MC}$ is the geometric increment $\Delta\mathbf{v}$ in the velocity vector of the point over the time interval Δt. The vector $\overrightarrow{AB}$ (directed along the opposite side of the parallelogram *MABC*) may be regarded as the same vector $\Delta\mathbf{v}$, but now carried to the point *A*.

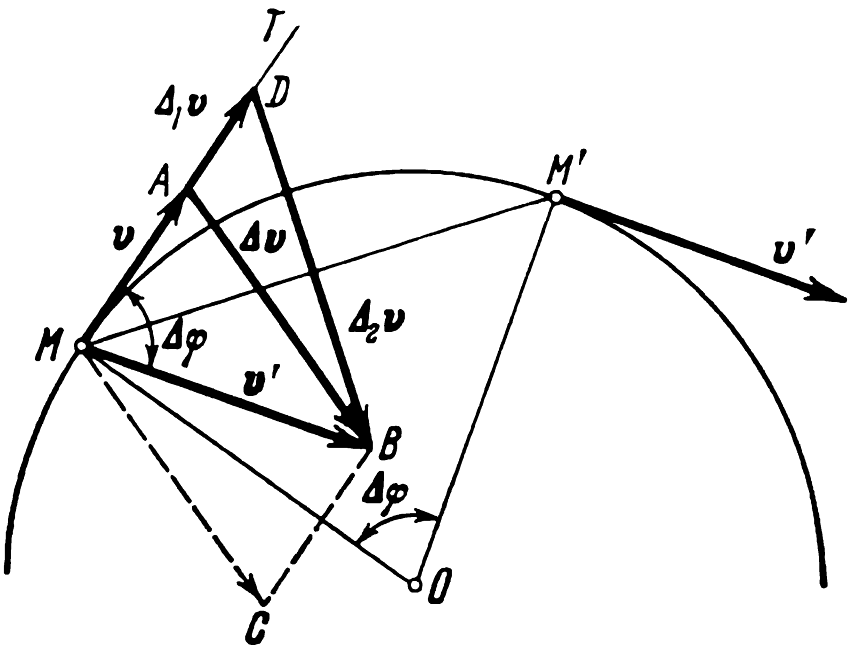


FIG. 151

Lay off a line segment $MD = MB = v'$ along the tangent to the trajectory at the point *M*, that is, in the direction of the velocity vector **v**. Then the numerical value of the vector $\overrightarrow{AD}$ will be $AD = MD - MA = v' - v = \Delta v$, that is, equal to the increment in the numerical value of the velocity **v** of the point over the time interval Δt.

Now decompose the vector $\Delta\mathbf{v} = \overrightarrow{AB}$ into two components: $\Delta_1\mathbf{v} = \overrightarrow{AD}$ and $\Delta_2\mathbf{v} = \overrightarrow{DB}$, of which the first is known in magnitude and direction. We find the magnitude and direction of the second vector $\Delta_2\mathbf{v}$ by joining points *D* and *B* with a vector. From the vector triangle *ADB* it follows that $\Delta\mathbf{v} = \Delta_1\mathbf{v} + \Delta_2\mathbf{v}$.

As follows from formula (69), the acceleration of the point at time t is

$$\mathbf{a}=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta_1\mathbf{v}+\Delta_2\mathbf{v}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta_1\mathbf{v}}{\Delta t}+\lim_{\Delta t\to 0}\frac{\Delta_2\mathbf{v}}{\Delta t}.$$

We now denote the vectors $\lim\limits_{\Delta t\to 0}\frac{\Delta_1\mathbf{v}}{\Delta t}$ and $\lim\limits_{\Delta t\to 0}\frac{\Delta_2\mathbf{v}}{\Delta t}$ that appear in the right-hand member of the last equation by $\mathbf{a}_t$ and $\mathbf{a}_n$ and seek their directions and numerical values. We begin with the vector

$$\mathbf{a}_t=\lim_{\Delta t\to 0}\frac{\Delta_1\mathbf{v}}{\Delta t}.$$

The direction of the given vector does not depend on the size of the time interval Δt. It always has the same direction as the vector $\Delta_1\mathbf{v}$: along the tangent to the trajectory of motion at the given position of the point, whence *the component* $\mathbf{a}_t$ *is termed the tangential acceleration of the point.*

Since the numerical value of the vector $\Delta_1\mathbf{v}$ is equal to the increment in the numerical value of the velocity $\mathbf{v}$ of the point over the time interval Δt, it follows that the numerical value of the tangential acceleration

$$a_t=\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}.$$

Since the numerical value v is a function of the time, and Δv is an increment in that function, it follows that

$$\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}=\frac{dv}{dt}.$$

Thus,

$$a_t=\frac{dv}{dt}. \tag{70}$$

The numerical value of the tangential acceleration of a point is equal to the derivative of the numerical value of the velocity with respect to time.

If in the course of time the magnitude of the velocity increases (the point is in accelerated motion), then the derivative dv/dt is positive and the tangential acceleration

$\mathbf{a}_t$ is along a tangent in the direction of the motion of the point. If in the course of time the magnitude of the velocity decreases (the point slows down), then the derivative dv/dt is negative and the tangential acceleration $\mathbf{a}_t$ is along a tangent in the direction opposite to that of the velocity of the point.

We will now define the second component of the acceleration of a point, that is,

$$\mathbf{a}_n = \lim_{\Delta t \to 0} \frac{\Delta_2 \mathbf{v}}{\Delta t}.$$

The direction of this vector clearly coincides with the limiting position of the vector $\Delta_2\mathbf{v}/\Delta t$ and, hence,* with the limiting position of the vector $\Delta_2\mathbf{v}$.

Take the isosceles triangle MBD (Fig. 151) and find the angle between the direction of the vector $\Delta_2\mathbf{v}$ and the tangent T to the trajectory of the point:

$$\widehat{(\Delta_2\mathbf{v}, T)} = \angle MDB = \frac{180^\circ - \Delta\varphi}{2} = 90^\circ - \frac{\Delta\varphi}{2}.$$

If the time interval Δt under consideration approaches zero without bound, then the point M' will also approach M without bound and the angle $\Delta\varphi$ between the tangents to the curve at these points will tend to zero. Consequently, the angle

$$\widehat{(\mathbf{a}_n, T)} = \lim_{v' \to v} \widehat{(\Delta_2\mathbf{v}, T)} = \lim_{\Delta\varphi \to 0} \left(90^\circ - \frac{\Delta\varphi}{2}\right) = 90^\circ.$$

The straight line perpendicular to the tangent to the curve at a given point is called the *normal to the curve at that point*. For this reason, the *component* $\mathbf{a}_n$ of acceleration of a point is called the *normal acceleration*. As is directly evident from Fig. 151, the *normal acceleration is always directed along the normal towards the concavity of the trajectory*. When the point is in motion along the circumference of a circle, the normal acceleration is directed along a radius to the centre of the circle, for which reason it is sometimes called the *centripetal acceleration.*

---

* If the vector is divided by a positive scalar Δt, the direction of the vector remains unchanged.

Let us now find the magnitude a_n of the normal acceleration of a point.

The triangles MBD and $OM'M$ are similar (Fig. 151) since both are isosceles and have equal vertex angles M and O ($\angle BMD = \angle MOM' = \Delta\varphi$, as angles with mutually perpendicular sides). From the similarity of these triangles we have

$$\frac{DB}{MM'} = \frac{MB}{OM'}.$$

In this proportion, $DB = |\Delta_2\mathbf{v}|$, which is the magnitude of the vector $\Delta_2\mathbf{v}$; MM' is the magnitude of the displacement vector $\overrightarrow{MM'}$ of the point during the time interval Δt, $MB = v'$ is the magnitude of the velocity of the point at time $t + \Delta t$, and $OM' = R$ is the radius of the circle.

Substituting the given values into the preceding proportion, we get

$$\frac{|\Delta_2\mathbf{v}|}{MM'} = \frac{v'}{R},$$

whence

$$|\Delta_2\mathbf{v}| = \frac{MM' \times v'}{R}.$$

The desired magnitude a_n of the normal acceleration of the point is

$$a_n = \lim_{\Delta t \to 0} \frac{|\Delta_2\mathbf{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{MM' \times v'}{R\,\Delta t} = \lim_{\Delta t \to 0} \left(\frac{1}{R} \times \frac{MM'}{\Delta t} \times v'\right).$$

The constant quantity $1/R$ may be taken outside the limit sign. Now the limit of a product is equal to the product of the limits involved (if they exist). Therefore,

$$a_n = \frac{1}{R} \lim_{\Delta t \to 0} \frac{MM'}{\Delta t} \lim_{\Delta t \to 0} v'.$$

In accordance with the definition of velocity [see Sec. 55, formula (63)], $\lim\limits_{\Delta t \to 0} \frac{MM'}{\Delta t} = v$, which is to say that this limit is equal to the magnitude of the velocity of the point.

Since as $\Delta t \to 0$ the vector $\mathbf{v}'$ approaches the vector $\mathbf{v}$ without bound, it follows that $\lim_{\Delta t \to 0} v' = v$. Substituting the given values into the preceding expression for a_n, we obtain

$$a_n = \frac{1}{R} \times v \times v = \frac{v^2}{R}.$$

The expressions for the tangential and normal accelerations of a point that were found for the special case of motion of a point on the circumference of a circle hold true also for the motion of a point along any trajectory; all we need to do is introduce the concepts of a circle and the radius of curvature of a curve at the given point.

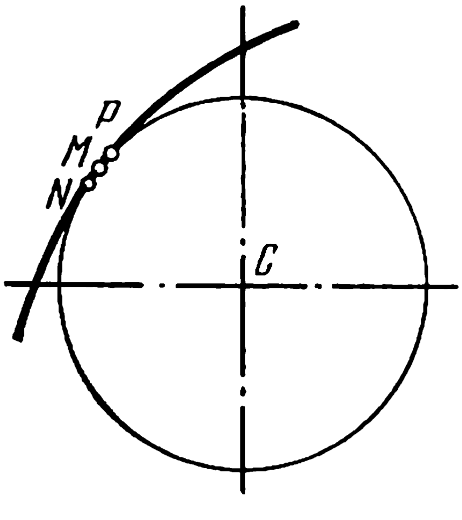


FIG. 152

Referring to Fig. 152, take a point M on a curve and also two points N and P on both sides of M and near M. We know that it is always possible to draw a circle (and only one circle) through three points M, N and P. The closer we take the points N and P to the point M, the less the element NMP of the curve will differ from the corresponding arc of the circle. In the limit, when the points N and P approach M without bound, the infinitesimal element NMP of the curve will coincide with the arc of the circle and will have a common tangent at the point M.

A circle that passes through a given point M and two other infinitely close points of the curve is called an osculating circle or circle of curvature of the curve at the point M. The radius ρ (the Greek letter rho) *of this circle drawn for the given point of the curve is termed the radius of curvature of the curve at that point. The centre of the circle is called the centre of the curve at the given point M.*

The quantity k, the inverse of the radius of curvature of the curve at the given point, is called the curvature of the curve at that point:

$$k = \frac{1}{\rho}. \qquad (71)$$

The smaller the curvature of the curve at the given point, the greater its radius of curvature ρ and the smaller its curvature k.*

Any sufficiently small portion of any curvilinear trajectory may be replaced by an arc of the corresponding circle and, hence, this curvilinear trajectory may be imagined as consisting of arcs of circles described by different radii and from different centres. From this it follows that to determine the normal acceleration of a point in any curvilinear motion it is possible to use the above-established formula if for the radius of the circle we substitute the radius of curvature of the trajectory at the appropriate point. We can now state the formula for the magnitude a_n of normal acceleration as follows: *the normal acceleration of a point is equal to the square of the velocity of the point divided by the radius of curvature of the trajectory at the corresponding point on the curve*:

$$a_n = \frac{v^2}{\rho}. \tag{72}$$

When specifying the motion of a point by the natural method, we know both the trajectory of the point (and, hence, its radius of curvature at any point) and the equation of motion of the point on the given trajectory, $s = f(t)$. With these facts we can find the velocity of the point using formula (64), and then the tangential and the normal acceleration of the point using formulas (70) and (72).

Now if we know the projections of the vector a on two mutually perpendicular axes (the tangent and the normal to the given point of the trajectory), it is easy to find both the magnitude and the direction of the acceleration vector **a** of the point:

$$\left.\begin{aligned} &a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}, \\ &\cos(\widehat{\mathbf{a}, \mathbf{a}_t}) = \frac{a_t}{a}, \quad \cos(\widehat{\mathbf{a}, \mathbf{a}_n}) = \frac{a_n}{a}. \end{aligned}\right\} \tag{73}$$

---

* For a circle, the radius of curvature ρ and the curvature k are constant and equal, respectively, to the radius of the circle, r, and the quantity $1/r$. For a straight line, which may be regarded as the arc of a circle of infinite radius, the radius of curvature is $\rho = \infty$ and the curvature is $k = 1/\rho = 0$.

If the point is in nonuniform rectilinear motion, its velocity varies in magnitude. In that case, $a_t = dv/dt \neq 0$. Now the radius of curvature of the trajectory (a straight line) is $\rho = \infty$ and $a_n = v^2/\rho = 0$. This means that *in nonuniform rectilinear motion, a point only has a tangential acceleration.*

If a point is in uniform curvilinear motion, then its velocity varies only in direction. Here, $v =$ constant and $a_t = dv/dt = 0$. The radius of curvature of the trajectory, $\rho \neq \infty$ and $a_n = v^2/\rho \neq 0$. This means that *in the case of uniform curvilinear motion, the point has only a normal acceleration.*

Now, if a point is in uniform rectilinear motion, then its velocity remains unchanged in magnitude and direction. In this case, $v =$ constant and $\rho = \infty$. Hence, the point will have no tangent and no normal acceleration. The acceleration is $a = \sqrt{a_t^2 + a_n^2} = 0$.

The foregoing special cases of motion of a point show that *tangential acceleration characterizes the change of its velocity in numerical value, whereas normal acceleration characterizes the change of velocity of the point in direction.*

Problem 61. A point is in motion around the circumference of a circle of radius $r = 4$ m via the law $s = 4.5t^3$ (s in metres, t in seconds). Find the magnitude a of the acceleration of the point and the angle α between the acceleration and velocity at time T, when the velocity is 6 m/s.

Solution. The magnitude of the velocity of the point is

$$v = \frac{ds}{dt} = \frac{d}{dt}(4.5t^3) = 13.5t^2 \text{ m/s}.$$

It is given that when $t = T$ the velocity of the point $v = 6$ m/s. Substituting these figures into the expression for velocity, we find $6 = 13.5T^2$, whence $T = \sqrt{\frac{6}{13{\cdot}5}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$ s.

The tangential acceleration of the point is

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(13.5t^2) = 27t \text{ m/s}^2.$$

For $t = T = \frac{2}{3}$ s, $a_t = 27 \times \frac{2}{3} = 18$ m/s^2.

The normal acceleration of the point is $a_n = v^2/\rho$. Substituting into this expression $v = 6$ m/s and $\rho = r = 4$ m, we get $a_n = 6^2/4 = 9$ m/s². The magnitude of the total acceleration of the point $a = \sqrt{a_t^2 + a_n^2} = \sqrt{18^2 + 9^2} = 9\sqrt{5}$ m/s². The angle α (see Fig. 153)

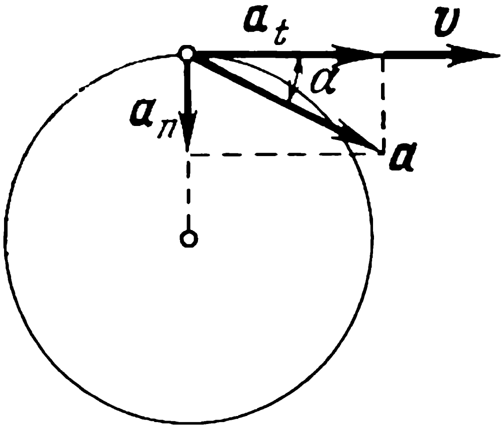


FIG. 153

between the acceleration **a** and the velocity **v** of the point is found from the formula $\tan\alpha = \frac{a_n}{a_t} = \frac{9}{18} = \frac{1}{2}$, whence $\alpha = \arctan\frac{1}{2} = 26°34'$.

Sec. *60. Determining the acceleration of a point from the equations of its motion in rectangular coordinates

Suppose a point is in plane motion in accord with specified equations:

$$x = f_1(t), \quad y = f_2(t).$$

Note positions M and M' (Fig. 154) of a moving point at times t and $t + \Delta t$. Denote by **v** and **v**′ the velocities of the point corresponding to these positions.

The projections of these vectors on the x-axis will be denoted by v_x and v'_x. Carrying the origin of the vector **v**′ to the point M, we will construct (as indicated in the preceding section) a vector $\Delta\mathbf{v}$—the increment in velocity of point M during time Δt:

$$\overrightarrow{MC} = \Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}.$$

But the projection of a geometric sum of vectors on some one axis is equal to the algebraic sum of the projections, on that axis, of the components of the vectors:

$$(\Delta \mathbf{v})_x = v'_x - v_x = \Delta v_x.$$

Hence, the projection on the x-axis of the vector Δx (velocity-increment vector) of the point is equal to the increment Δv_x in the projection of velocity of the point on that axis.

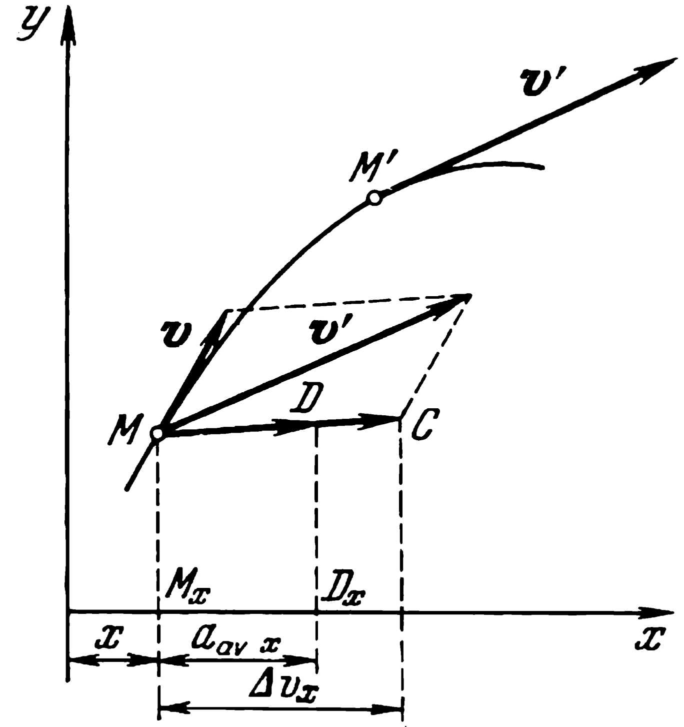


FIG. 154

The vector $\overrightarrow{MD}$ of average acceleration of the point* during the time interval Δt is, by (68), equal to

$$\overrightarrow{MD} = \mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t}.$$

The projection of this vector on the x-axis is

$$M_x D_x = a_{\text{av}\ x} = \frac{(\Delta \mathbf{v})_x}{\Delta t} = \frac{\Delta v_x}{\Delta t}.$$

Passing to the limit as $\Delta t \to 0$ and denoting the projection of the true acceleration of the point on the x-axis by a_x, we obtain

$$a_x = \lim_{\Delta t \to 0} a_{\text{av}\ x} = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}.$$

The projection v_x of the velocity of the point on the x-axis (that is, the velocity of the projection of the given point on that axis) is a function of the time t; Δv_x is the increment in this function that corresponds to the increment Δt in the argument t, and $\lim\limits_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$ is the derivative of the given function.

---

* In Fig. 154, the vector $\mathbf{a}_{\text{av}}$ is depicted by the vector $\overrightarrow{MD}$. Of course the magnitude of this vector may even be greater than the magnitude of the vector $\overrightarrow{MC}$ (if $\Delta t < 1$).

Consequently, $a_x = dv_x/dt$. But the projection of the velocity of the point on the coordinate axis is itself equal (see Sec. 57) to the derivative of the corresponding coordinate of the point with respect to time, $v_x = dx/dt$.

Thus,

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}.$$

Reasoning in a similar manner, we can find the projection of the acceleration of the point on the y-axis:

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt}.$$

The projections of the acceleration of a point on fixed axes are equal to the first derivatives, with respect to time, of the projections of the velocity of the point on the corresponding coordinate axes or to the second derivatives, with respect to time, of the appropriate coordinates of the point:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \tag{74}$$

During the displacement of the point during time Δt along an arc of the trajectory from position M to position M', the velocity of the projection of this point on the x-axis receives an increment Δv_x. Hence, dv_x/dt determines the acceleration of the projection of the given point on the x-axis.

Thus, *the projection of the acceleration of a point on a coordinate axis determines the magnitude and the direction of the acceleration of that point on the same axis and at the same instant of time.*

From the projections of the acceleration vector on the coordinate axes we find the magnitude and direction of the vector. The magnitude of acceleration of the point is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \tag{75}$$

The direction of the acceleration vector is found from the formulas

$$\cos(\widehat{\mathbf{a},\ x})=\frac{a_x}{a}, \quad \cos(\widehat{\mathbf{a},\ y})=\frac{a_y}{a}. \tag{76}$$

If a point is in random motion in space (instead of in a plane), then, if we know the third equation of motion of the point, $z = f_3(t)$, we can in a similar fashion find the projection a_z of the acceleration of the point on the third coordinate axis, and then also the magnitude of the acceleration vector of the point and its direction in space.

Problem 62. Write down the equations of motion in rectangular coordinates and find the velocity and acceleration of the end point M of a crank OM rotating about a fixed centre O (Fig. 155). The crank length is $OM = r$. The angle of rotation of the crank relative to the horizontal axis varies in accord with the law $\varphi = \omega t$.

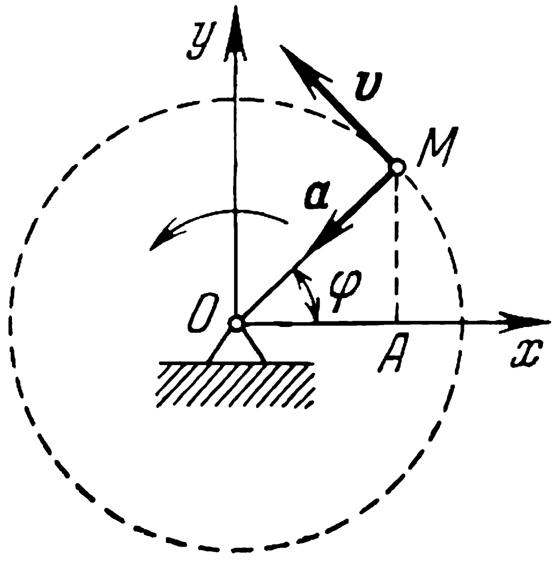


FIG. 155

Solution. Take a system of coordinate axes x and y with origin at O. The coordinates of the point M in this system (see Fig. 155) is

$$x = OA = OM \times \cos\varphi = r\cos\omega t,$$

$$y = AM = OM \times \sin\varphi = r\sin\omega t.$$

The projections of the velocity of point M on the coordinate axes are

$$v_x=\frac{dx}{dt}=-r\omega\sin\omega t, \quad v_y=\frac{dy}{dt}=r\omega\cos\omega t.$$

The magnitude of the velocity is

$$v=\sqrt{v_x^2+v_y^2}=\sqrt{(-r\omega\sin\omega t)^2+(r\omega\cos\omega t)^2}=r\omega=\text{constant}.$$

The velocity of point M is constant in absolute value. The direction of the velocity vector $\mathbf{v}$ of the point may be found from formula (67):

$$\cos(\widehat{\mathbf{v}, x}) = \frac{v_x}{v} = -\frac{r\omega \sin \omega t}{r\omega} = -\sin \omega t = -\sin \varphi,$$

$$\cos(\widehat{\mathbf{v}, y}) = \frac{v_y}{v} = \frac{r\omega \cos \omega t}{r\omega} = \cos \omega t = \cos \varphi.$$

The projections of the acceleration vector of the point M on the coordinate axes are

$$a_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt} = -r\omega^2 \cos \omega t,$$

$$a_y = \frac{d^2y}{dt^2} = \frac{dv_y}{dt} = -r\omega^2 \sin \omega t.$$

The magnitude of acceleration is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-r\omega^2 \cos \omega t)^2 + (-r\omega^2 \sin \omega t)^2} = r\omega^2 = \text{constant}.$$

The acceleration of M is constant in absolute value. It is not equal to zero despite the fact that the velocity of the point is a constant. Due to the curvilinear nature of the trajectory of the point, the direction of the velocity changes all the time.

The direction of the acceleration vector $\mathbf{a}$ of M may be found from formulas (76):

$$\cos(\widehat{\mathbf{a}, x}) = \frac{a_x}{a} = -\frac{r\omega^2 \cos \omega t}{r\omega^2} = -\cos \omega t = -\cos \varphi,$$

$$\cos(\widehat{\mathbf{a}, y}) = \frac{a_y}{a} = -\frac{r\omega^2 \sin \omega t}{r\omega^2} = -\sin \omega t = -\sin \varphi.$$

It is easy to see that in the case of uniform motion around a circle, the acceleration $\mathbf{a}$ of M is directed along the radius r to the centre of the circle.

Problem 63. A point is in motion along a helical curve in accordance with the equations

$$x = 2 \cos 4t, \qquad x = 2 \sin 4t, \qquad z = 2t$$

(x, y, z in metres, t in seconds). Find the numerical values of the velocity and acceleration of the point and also the radius of curvature of its trajectory.

Solution. The projections of the velocity of the coordinate axes are

$$v_x = \frac{dx}{dt} = -8 \sin 4t, \quad v_y = \frac{dy}{dt} = 8 \cos 4t, \quad v_z = \frac{dz}{dt} = 2.$$

The magnitude of the velocity is

$$v=\sqrt{v_x^2+v_y^2+v_z^2}=\sqrt{(-8\sin 4t)^2+(8\cos 4t)^2+2^2}$$
$$=\sqrt{64(\sin^2 4t+\cos^2 4t)+4}=\sqrt{68}=2\sqrt{17}\ \text{m/s}=\text{constant}.$$

The projections of the acceleration on the coordinate axes are

$$a_x=\frac{dv_x}{dt}=-32\cos 4t,\qquad a_y=\frac{dv_y}{dt}=-32\sin 4t,\qquad a_z=\frac{dx_z}{dt}=0.$$

The magnitude of the acceleration is

$$a=\sqrt{a_x^2+a_y^2+a_z^2}=\sqrt{(-32\cos 4t)^2+(-32\sin 4t)^2}=32\ \text{m/s}^2.$$

Since the magnitude v of the velocity is a constant, the tangential acceleration $a_{\text{t}}=dv/dt=0$. Thus the acceleration $\mathbf{a}$ of the point consists solely of its normal acceleration: $a=a_{\text{n}}=v^2/\rho$. From this we find the radius of curvature of the trajectory:

$$\rho=\frac{v^2}{a_{\text{n}}}=\frac{v^2}{a}=\frac{(2\sqrt{17})^2}{32}=\frac{17}{8}=2\frac{1}{8}\text{m}.$$

CHAPTER XIII

SPECIAL CASES OF THE MOTION OF A POINT

Sec. 61. Uniform motiont of a point

As we know, the uniform motion of a point is motion in which the point covers equal distances in equal intervals of time.

The magnitude (absolute value) of the velocity of the point in this motion $v = s/t =$ constant. From this it follows that in such motion

$$S = vt. \tag{77}$$

The path covered by a point in uniform motion (and only in such motion) is equal to the product of the velocity by the time. Using formula (77), we can find any one of three quantities S, v and t in the case of uniform motion when the other two are known.

If at the initial time the point is located at a distance from the origin of a coordinate system, s_0, then the *distance of the point from the origin* is, *at time* t, equal to

$$s = s_0 + vt. \tag{78}$$

This equation defines the *law of uniform motion of a point.* Since the distance of the point from the origin varies linearly with time, the graph of the motion of the point [the graph of the relationship $s = f(t)$] is a straight line.

From the graph of the motion of the point we can find not only the distance of the point from the origin but also the velocity of motion. Referring to Fig. 156, suppose the graph of a given uniform motion of a point is a straight line inclined at an angle of α to the time axis. Also, assume that in constructing the graph we used the following scales: time μ_t s/mm, that is, 1 mm on the time axis corresponds to μ_t seconds, and distance μ_s m/mm, that is, 1 mm on the distance scale corresponds to μ_s metres, Note on the time axis a time interval $t = OA \times \mu_t = DC \times \mu_t$. This time interval is associated with the path S that is covered: $S = \mu_s (s - s_0) = \mu_s (AB - AC) = \mu_s \times BC$. Then the velocity of uniform motion of the point is

$$v = \frac{S}{t} = \frac{BC}{DC} \times \frac{\mu_s}{\mu_t} = \frac{\mu_s}{\mu_t} \tan\alpha.$$

The velocity of uniform motion of a point is numerically equal to the tangent of the angle of inclination of a rectilinear graph of its motion to the time axis multiplied by the ratio of the distance scale to the time scale:

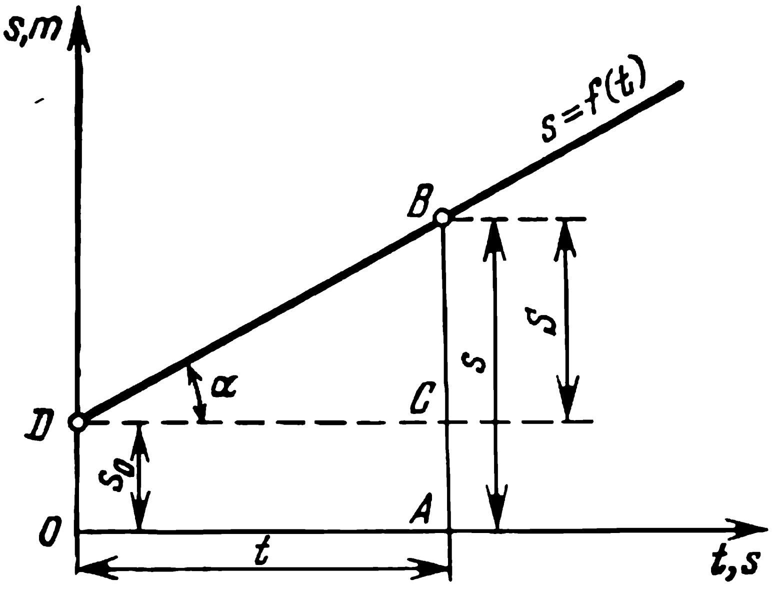


FIG. 156

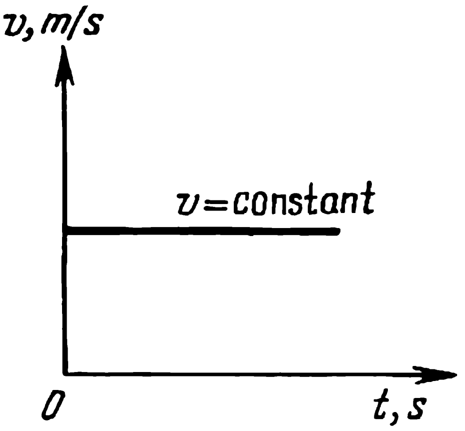


FIG. 157

$$v = \frac{\mu_s}{\mu_t} \tan\alpha. \tag{79}$$

Just as we constructed a distance graph, so we can construct, in appropriate scales, the graph of velocity of every motion of a point, that is, the graph of the function $v = f'(t)$.

Since the velocity of a point in the uniform motion is a constant, the graph of velocity of uniform motion is a straight line parallel to the time axis (Fig. 157).

If a point is in curvilinear uniform motion, it has (as we said in Sec. 59) only a normal acceleration $a = a_n = v^2/\rho$. But if a point is in rectilinear uniform motion, then it has no acceleration at all, $a = 0$.

Problem 64. The graph of motion of a point is a straight line (Fig. 158) that intercepts a line segment $a = -10$ mm on the time axis and a segment $b = 25$ mm on the distance axis. The scales used in constructing the graph are: distance $\mu_s = 2$ km/mm and time $\mu_t = 0.2$ h/mm. Find the velocity of the point and write down the equation of motion of the point along the trajectory.

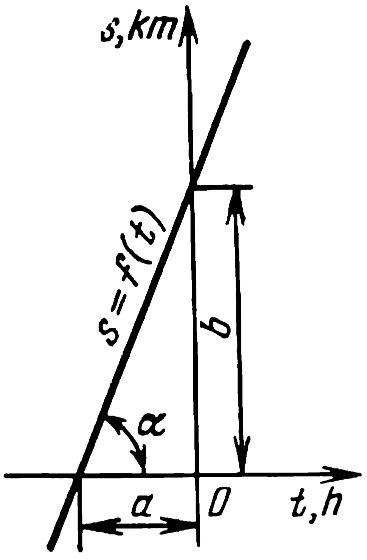


FIG. 158

Solution. Since the graph of motion of the point, that is, the graph of the function $s = f(t)$, is a straight line, the point is in uniform motion. By formula (79), the velocity of the point is

$$v = \frac{\mu_s}{\mu_t}\tan\alpha = \frac{b}{a}\times\frac{\mu_s}{\mu_t} = \frac{25}{10}\times\frac{2\ \mathrm{km/mm}}{0.2\ \mathrm{h/mm}} = 25\ \mathrm{km/h}.$$

Now write down the equation of motion of the point. For $t = 0$, the distance of the point from the origin of the reference system is $s_0 = b\mu_s = 25\ \mathrm{mm} \times 2\ \mathrm{km/mm} = 50$ km. Thus the equation of motion of the point takes the form $s = s_0 + vt = 50 + 25t$ (where s is expressed in kilometers and t in hours).

Sec. 62. Rectilinear nonuniform motion of a point

If for the x-axis we take the rectilinear trajectory of a point, then the equation of motion of the point along that trajectory assumes the form $s = x = f(t)$. The numerical value of the velocity is given by the formula

$$v = \frac{ds}{dt} = \frac{dx}{dt}. \tag{80}$$

The numerical value of the velocity of rectilinear motion of a point (along a coordinate axis) is equal to the derivative of the coordinate of the point with respect to time.

The velocity vector **v** of the point is always in the direction of its motion, that is, along the x-axis in our case. And if the derivative dx/dt is positive for a given value of t, the direction of velocity coincides with the positive direction of the x-axis. And if dx/dt is negative, the velocity is in the negative direction of the x-axis. In the case of rectilinear motion, the velocity of the point will vary only in magnitude and the point will only have tangential acceleration:

$$a = a_t = \frac{dv}{dt} = \frac{d^2S}{dt^2} = \frac{d^2x}{dt^2}. \qquad (81)$$

The numerical value of the acceleration of rectilinear motion of a point is equal to the first derivative of the numerical value of the velocity with respect to time or to the second derivative of the coordinate of the point with respect to time.*

If for a given value of t the second derivative d^2x/dt^2 is positive, then the acceleration vector **a** is in the positive direction of the x-axis; if d^2x/dt^2 is negative, the vector **a** is in the negative direction of the x-axis. Consequently if the derivatives dx/dt and d^2x/dt^2 have the same sign, the vectors **v** and **a** are in the same direction and the motion of the point is accelerated. If the derivatives dx/dt and d^2x/dt^2 have opposite signs, the motion is decelerated motion.

When an analytic investigation of the motion of a point is complicated or when the law of its motion is specified graphically, the velocity, acceleration and path covered by the point may be found graphically.

Referring to Fig. 159, suppose we know** the graph of motion of a point [the curve $s = f(t)$] and suppose we have taken the following scales in constructing the graph: time μ_t s/mm, that is, 1 mm on the time axis corresponds to μ_t seconds, and distance μ_s m/mm, or 1 mm on the distance axis corresponds to μ_s metres.

On the time axis take two close times t_1 and t_2. The distances of a point from the origin corresponding to these times are given on the graph of motion by the line segments AM_1 and BM_2. Taking into account the scales, we find that the time increment $\Delta t = t_2 - t_1 = M_1C \times \mu_t$ is associated with the following increment in distance of the moving point:

---

* Formula (81) holds true only for rectilinear motion of a point. In the case of curvilinear motion, the point will of necessity have a normal acceleration, and its total acceleration will not be equal to the first derivative of the numerical value of velocity with respect to time or to the second derivative of the path with respect to time.

** The graph of the function $s=f(t)$ may be constructed in different ways (see Sec. 54.1). Since in the case of nonuniform motion of a point, $s = x$ may be any function of t, it follows that the graph of such motion is a certain curve.

$$\Delta s = (M_2B - M_1A)\,\mu_s = M_2C \times \mu_s.$$

Hence the numerical value of the average velocity of the point over the time interval Δt is

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{M_2C}{M_1C} \times \frac{\mu_s}{\mu_t} = \frac{\mu_s}{\mu_t} \tan\alpha',$$

where $\tan\alpha'$ is the tangent of the angle between the secant, drawn through the points of the graph of motion that correspond to the beginning and end of the given time interval, and the positive direction of the time axis.

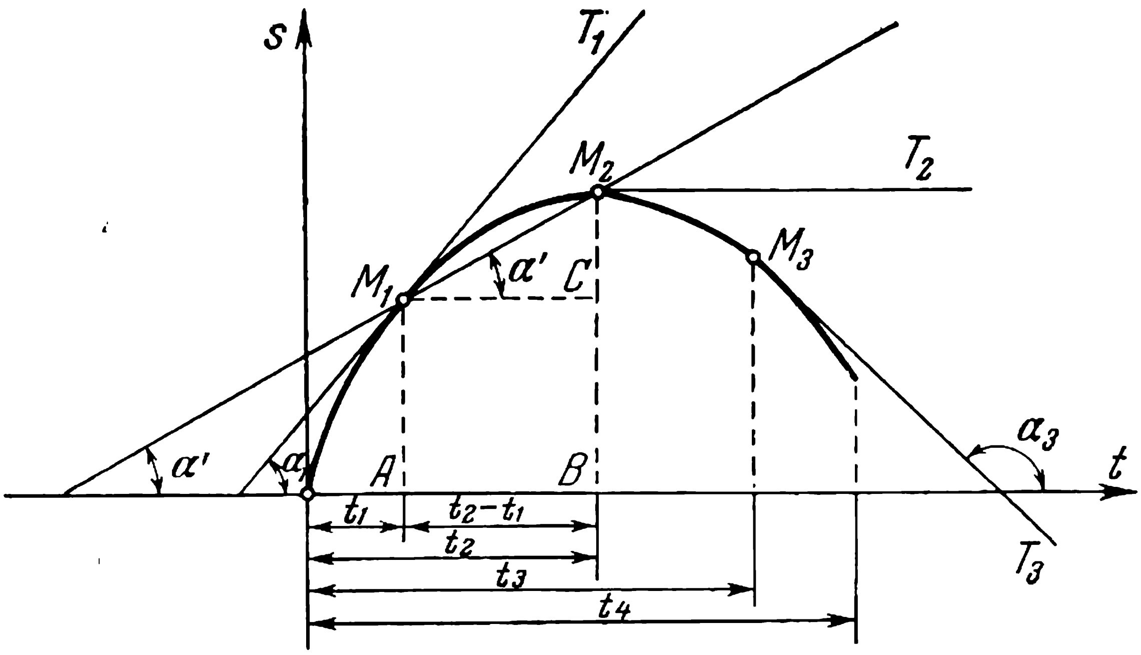


FIG. 159

To find the velocity of a point at some time t, we allow the time interval $\Delta t = t_2 - t_1$ to go to zero so that time t_2 approaches the time t_1 without bound. Then point M_2 will approach without bound (on the graph of motion) the point M_1, and in the limit the secant M_1M_2 will turn into the tangent line M_1T_1, and the angle α' will turn into the angle α_1 (Fig. 159). Thus

$$v = \lim_{\Delta t \to 0} v_{\text{av}} = \lim_{\Delta t \to 0} \frac{\mu_s}{\mu_t} \tan\alpha' = \frac{\mu_s}{\mu_t} \tan\alpha_1.$$

The numerical value of the velocity v at any time is equal to the tangent of the angle α between the positive time axis *and the tangent to the graph of motion at the appropriate point multiplied by the ratio of the distance scale to the time scale*:

$$v = \frac{\mu_s}{\mu_t} \tan \alpha. \qquad (82)$$

The formula (82) permits determining not only the magnitude of the velocity but also its direction. The velocity of a point will be directed along the tangent to the trajectory in the direction of positive distances (increasing s) for a positive value of tan α and in the opposite direction in the case of a negative value of tan α.

Referring to Fig. 159, we see that time t_1 is associated with an increase in the distance s of the point from the origin of the reference system as the time increases. The angle α_1 between the tangent to the appropriate point of the graph and the positive direction of the t-axis is acute; hence tan $\alpha_1 > 0$ and $v_1 > 0$. The angle α_1 will remain acute and will diminish constantly in magnitude for every instant of time in the interval from zero to time t_2. Therefore in the course of this time interval, the point performs decelerated motion in the positive direction of distances from the origin. At time t_2, the tangent to the graph is parallel to the t-axis and the angle $\alpha = 0$, hence, tan $\alpha = 0$ and $v_2 = 0$. At time t_2 the velocity of the point passes through zero and the point reverses its direction of motion. The angle α_3 between the tangent to the curve at the point M_3, which corresponds to time t_3, and the t-axis is obtuse; hence, tan $\alpha_3 < 0$ and $v_3 < 0$. In the interval between t_2 and t_4, the angle α is obtuse and constantly decreases, while the tangent of the angle α is negative and increases in absolute value. Therefore in the course of this time interval the point performs accelerated motion in the opposite direction to positive distances from the origin.

As we see, the graph of motion permits not only finding the velocity of a point at any time but also gives us some idea of the nature of the motion over the entire time interval.

In order to get a more vivid picture of the nature of change in velocity of a point and to be able to determine its acceleration graphically, we have to construct a graph of the velocity, that is, a graph of the function $v = f'(t)$. To construct it, take two mutually perpendicular coordinate axes and on the axis of abscissas lay off (to some scale μ_t) the values of time t, and on the axis of ordinates lay off the numerical values of velocity v of the point to some scale μ_v.

If for various times t we know* the corresponding values of velocity v, we can construct a number of points using the coordinates of t and v. Joining these points with a smooth line, we obtain the curve shown in Fig. 160, which is the graph of the function $v = f'(t)$.

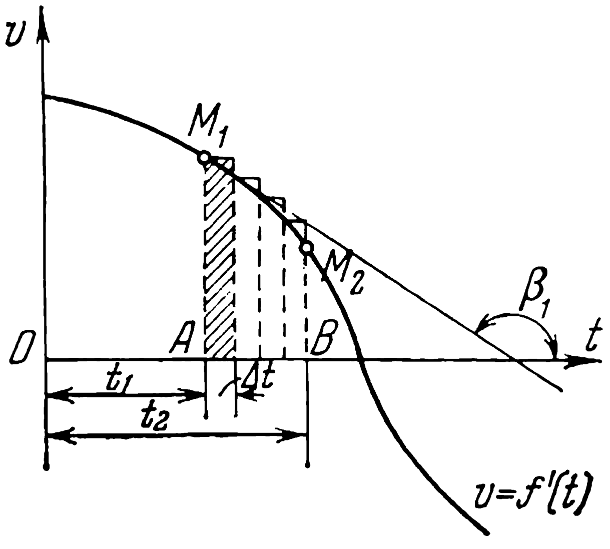


FIG. 160

Since by formula (70) the numerical value of the tangential acceleration of a point $a_t = \lim\limits_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$, where Δv is the increment in the absolute value of velocity, it follows that by reasoning as we did when determining the velocity of the point graphically we arrive at a similar conclusion: *the numerical value of a_t, the tangential acceleration of a point, is equal at any instant of time to the tangent of the angle β between the positive time axis and the tangent to the graph of velocity at the corresponding point multiplied by the ratio of the velocity scale to the time scale*:

$$a_t = \frac{\mu_v}{\mu_t} \tan \beta. \tag{83}$$

Formula (83) determines the tangential acceleration of a point not only in magnitude but also in direction. Thus, for time t_1, the angle $\beta = \beta_1$ is obtuse (Fig. 160) and therefore $\tan \beta_1 < 0$ and $a_t < 0$. Consequently, at that instant the acceleration a_t is directed along the tangent to the trajectory in the direction opposite the motion of the point.

---

* The values of velocity for different instants of time may be found from the analytic relationship $v = f'(t)$, if it is known, or graphically, as indicated earlier.

In the case of rectilinear motion, the normal acceleration $a_n = 0$ and for that reason formula (83) gives the total acceleration $a = a_t$ of the point for such motion.

Determining the velocity $v = ds/dt$ graphically from the curve of motion $s = f(t)$, and the tangential acceleration $a_t = dv/dt$ from the velocity curve $v = f'(t)$ is frequently called graphical differentiation of the function at hand.

Knowing the velocity graph of a point, it is possible to find (graphically) the path S covered by the point over some time interval. Suppose it is required (see Fig. 160) to find, from the velocity curve, the path covered by a point during the time interval $t_2 - t_1$. Divide the whole time interval into a large number n of small intervals Δt. Since the interval Δt is small, the motion of the point during this time may be regarded as uniform, i.e., as having the same velocity v as the point had at the start of the interval. Then the path covered by the point is $\Delta S \approx v\Delta t$, or is approximately equal to the cross-hatched area of the rectangle in Fig. 160. The larger the number n of subintervals the given time interval $t_2 - t_1$ is partitioned into, the smaller the subinterval Δt and the greater the accuracy with which we can regard the motion of the point during the time of the subinterval as being uniform. If n tends to infinity, then Δt will tend to zero and the length S of the path covered by the point during time $t_2 - t_1$ will be equal to the limit of the sum of areas of an infinite number of elementary rectangles into which the curvilinear figure AM_1M_2B is partitioned, in other words, it will be equal to the area of the figure (to some scale).

The scale in which the area of the [figure AM_1M_2B expresses the path length covered by the point depends on the scales of velocity and time used in constructing the velocity curve. For instance, suppose the velocity curve was constructed for convenience in computing the area of a curvilinear figure on millimetre paper and the area of the figure is given in square millimetres. Also suppose the velocity curve was constructed with the use of the velocity scale μ_v (m/s)/mm, which means 1 mm of the axis of ordinates depicts a velocity in μ_v m/s, and the time μ_t s/mm, which means 1 mm of the axis of abscissas depicts a time interval of μ_t s. Then the path S covered by the point is expressed in metres in the following manner:

$$S = \text{area } AM_1M_2B \text{ mm}^2 \times \mu_v \frac{\text{m}}{\text{s}\cdot\text{mm}} \times \mu_t \frac{\text{s}}{\text{mm}}$$

$$= \mu_v\mu_t \times \text{area } AM_1M_2B \text{ m},$$

where area AM_1M_2B is given in square millimetres.

Thus, *the path S covered by a point during a certain time interval is equal to the area of the figure bounded by the velocity curve, the time axis and the ordinates corresponding to the start and finish of the given time interval multiplied by the product of the scales of velocity and time*:

$$S = \mu_v\mu_t \times \text{area } AM_1M_2B. \tag{84}$$

Since the velocity of a point is given by the formula $v = dS/dt$ and, hence, the path covered is given by the formula $S = \int v\,dt$, determining the path S covered by a point on the basis of its velocity curve is frequently termed graphical integration. To get a clear-cut picture of the nature of varying acceleration during the motion of a point, we can construct a graph of the acceleration, that is, a graph of the function

$$a_t = f''(t).$$

It is constructed much like the graph of velocities, the only difference being that acceleration (to some scale μ_a) instead of velocity is laid off on the axis of ordinates.

Problem 65. Referring to Fig. 161*a*, a frame AB (a kind of link-gear member) rigidly joined to a guide bar DE is set in motion by a crank OC (located in front of the frame and guide bar). The crank rotates about point O uniformly, and the angle φ varies with time via the law $\varphi = \frac{\pi}{6}t$. The end C of the crank is hinged to a slide block that slides in a groove of the frame. The length of the crank $OC = a = 250$ mm. Determine the motion of the frame.

Solution. When the crank is turned, the frame AB performs oscillatory motion about point O, all points of the frame performing the same rectilinear motion. Let us consider the motion of some point of the frame, say the midpoint M. The trajectory of this point is a straight line that coincides with the axis DE.

Let us take the fixed point O (the axis of rotation of the crank) as the origin of the reference system for reckoning distance and write down the equation of motion of point M.

From the triangle OCM (Fig. 161a) we find $OM = OC \times \cos\varphi$. Denoting the distance of point M from the origin O by s and taking into account that (by hypothesis) $OC = a = 250$ mm and the angle $\varphi = \frac{\pi}{6}t$ radians, we have

$$s = a\cos\varphi = a\cos\frac{\pi}{6}t \text{ mm.} \qquad \text{(I)}$$

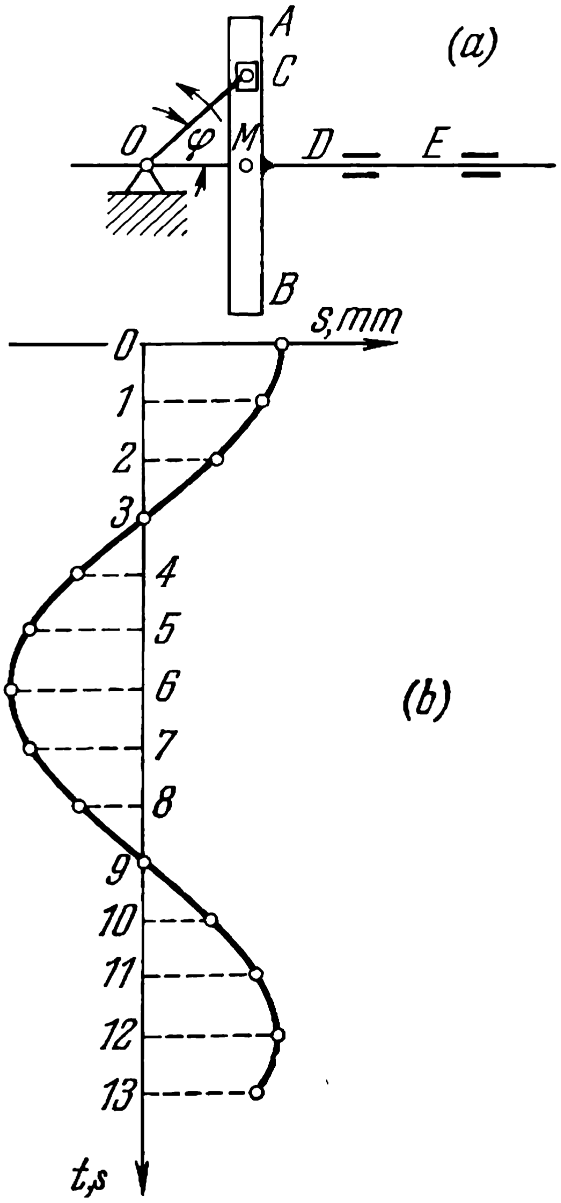


FIG. 161

The motion of a point moving in accord with such a law (that is, via the law $s = a\cos\omega t$ or $s = a\sin\omega t$) is termed *harmonic oscillatory motion.* The basic characteristics of such motion are: *the amplitude a of oscillation of a point, that is, its greatest deviation from the mean position, and the period T of oscillation, that is, the time of one complete oscillation of the point.*

Clearly, the point must be at the same distance s from the origin of the reference system at the beginning and at the end of the time interval T. For this purpose, it is necessary that during time T the argument of the cosine in equation (I) vary by the amount 2π, that is, that we have the following identity: $\cos\omega(t + T) \equiv \cos(\omega t + 2\pi)$, whence $T = \frac{2\pi}{\omega}$. In our example, $T = \frac{2\pi}{\pi/6} = 12$ s and $a = 250$ mm. Figure 161b shows the graph of motion of point M. The time t in seconds is laid off on the vertical axis and the distance s in mm is laid off on the horizontal axis.

The resulting curve is a cosine curve, which is to say, a sine curve displaced relative to the coordinate origin.

Sec. 63. Uniformly variable motion of a point

Uniformly variable (uniformly accelerated or uniformly decelerated) motion which is frequently encountered in engineering practice *is the motion of a point such that during* *equal and arbitrarily taken time intervals the magnitude of velocity of the point varies by the same amount.*

As we know, the variation of velocity of a point in magnitude is characterized by tangential acceleration. From this it follows that in the uniformly variable motion of a point the value of its tangential acceleration $a_t = dv/dt =$ constant. Therefore, separating the variables and integrating the equation, we have

$$dv = a_t dt, \quad v = \int a_t dt = a_t t + C_1.$$

The arbitrary constant C_1 is found from the initial conditions. Suppose at the initial time $t = 0$ the point had a velocity equal to v_0. Substituting $t = 0$ and $v = v_0$ into the preceding equation, we get $C_1 = v_0$. Hence the *formula for the velocity of a point in uniform variable motion* is of the familiar (from physics) form:

$$v = v_0 + a_t t, \tag{85}$$

where v is the magnitude of velocity of the point at the given time, v_0 is the magnitude of the initial velocity of the point, that is, the velocity at time $t = 0$, a_t is the algebraic value of the tangential acceleration of the point, which is positive in the case of accelerated motion and negative in the case of decelerated motion.

If $a_t < 0$, that is, if the tangential motion of the point is in the direction opposite that of the initial velocity of the point, then the motion is uniformly decelerated until the velocity v reaches zero, which, as is evident from formula (85), corresponds to the time $t = -v_0/a_t$. After that instant, if the direction of the tangential acceleration remains unchanged, the velocity v reverses its direction; its magnitude increases, the motion of the point is uniformly accelerated, and a_t becomes positive.

To determine the path S covered by the point in the case of uniformly variable motion, let us take advantage of the relationship (64): $v = dS/dt$, whence we have $dS = v\,dt = (v_0 + a_t t)\,dt = v_0 dt + a_t t\,dt$. Integrating this equation, we obtain

$$S = \int v_0\,dt + \int a_t t\,dt = v_0 t + \frac{a_t t^2}{2} + C_2.$$

The arbitrary constant C_2 is found from the initial conditions on the assumption that for $t = 0$ the path covered by the point is $S = 0$. Substituting these values of t and S into the above equation, we get $C_2 = 0$. Again we obtain a familiar physics *equation for the path of a point in the case of uniformly variable motion*:

$$S = v_0 t + \frac{a_t t^2}{2}. \tag{86}$$

In this formula, a_t is to be understood as the algebraic value of the tangential acceleration, that is, it has to be substituted into the formula with the appropriate sign.

Note that when calculating the path covered by a point, we can use formula (86) only on condition that the motion of the point is in one direction. If in the case of uniformly decelerated motion the velocity of the point passes through zero, then, as we have already seen, the tangential acceleration changes sign. Then the path covered by the point, $S = S_1 + S_2$, where S_1 is the path covered by the point during its uniformly decelerated motion, that is, during the time interval from $t = 0$ to $t = -v_0/a_t$, and S_2 is the path covered by the point during its uniformly accelerated motion after the velocity of the point passes through zero, that is, after time $t = -v_0/a_t$.

Formula (86) can also serve as the equation of uniformly variable motion of a point along its trajectory, that is, for determining the distance s of a point from the origin of the reference system for reckoning distance on the condition that the point is at the origin at the initial instant of time. But if at time $t = 0$ the point is not located at the origin but at some position away from the origin, determined by the distance s_0, then at time t the distance of the point from the origin will be $s = s_0 + S$.

Substituting into the last expression the value of the path S covered by the point, we obtain a general equation of uniformly variable motion of the point along its trajectory:

$$s = s_0 + v_0 t + \frac{a_t t^2}{2}. \tag{87}$$

The formulas (85), (86) and (87) equally hold true for both rectilinear and curvilinear uniformly variable motion of a point.

In the case of rectilinear motion of a point, the magnitude of its normal acceleration $a_n = v^2/\rho = 0$ and therefore its total acceleration $\mathbf{a} = \mathbf{a}_t$. For this reason, ordinarily, when applying formulas (85), (86) and (87) to rectilinear uniformly variable motion, the index t in the notation for acceleration is dropped.

Now in the case of curvilinear uniformly variable motion of a point, it is only the absolute value of the tangential component $\mathbf{a}_t$ of the acceleration $\mathbf{a}$ of the point that enters the above-mentioned formulas. In the case of any curvilinear motion of the point, the magnitude of the normal component of the acceleration is $a_n = v^2/\rho \neq 0$ and therefore the magnitude of the acceleration of the point is $a = \sqrt{a_t^2 + a_n^2} \neq a_t$.

An instance of uniformly variable motion of a point is the motion of a body in vertical flight under the action of the force of gravity. From physics we know that under the action of a constant force, a body receives a constant acceleration. If we disregard air resistance and the variation of the force of gravity with altitude, it may be assumed that the acceleration of a freely falling body or one thrown vertically upwards (this acceleration is usually denoted by the letter g) is constant. This acceleration varies with the geographic latitude and the altitude of the spot above sea level; but the variation is slight* and is ordinarily discarded. We assume the "normal" acceleration of a freely falling body to be $g = 9.81$ m/s². If in formulas (85) and (86) we take the acceleration $a_t = g$ and $S = h$, we get formulas for the vertical motion of a body due to the force of gravity:

$$v = v_0 \pm gt, \tag{88}$$

$$h = v_0 t \pm \frac{gt^2}{2}. \tag{89}$$

In these formulas, we take the plus sign in front of g in the case of free fall (uniformly accelerated motion) and the minus sign for a body thrown vertically upwards (uniformly decelerated motion).

---

* At a latitude of 45° the acceleration $g = 980.665$ cm/s² ≈ 9.81 m/s².

If the body begins to fall without an initial velocity, then $v_0 = 0$ and the above formulas take the form

$$v = gt \quad \text{and} \quad h = \frac{gt^2}{2}.$$

Eliminating t from these equations, we obtain

$$t = \frac{v}{g} \quad \text{and} \quad h = \frac{gv^2}{2g^2} = \frac{v^2}{2g},$$

whence the familiar *formula of Galileo*:

$$v = \sqrt{2gh}, \tag{90}$$

where v is the velocity of a body falling without an initial velocity from altitude h.

Since the formulas (88), (89) and (90) are derived from the conditions of motion of a body in a vacuum (in the absence of air resistance), they can be used in actual conditions only if the air resistance can be ignored, that is, when the weight of the body is great in comparison with the force of air resistance, and the height h is small.

Problem 66. A point with initial velocity $v_0 = 3$ m/s receives an acceleration equal in absolute value to 1 m/s^2 and in the opposite direction to the initial velocity of the point. Find the path covered by the point during time $t = 10$ s and its distance from the origin of the reference system at the end of the tenth second if at the initial time the point was located at a distance of $s_0 = 5$ m from the origin.

Solution. Assuming $v = 0$, from formula (85) $v = v_0 + a_t t$, where $a_t = -1$, we determine the time of passage of the velocity of the point through zero:

$$t = -\frac{v_0}{a_t} = -\frac{3}{-1} = 3\text{ s}.$$

Thus, during the first three seconds of its motion the point was in uniformly decelerated motion in the positive direction from the origin, and in the subsequent seven seconds the point was in uniformly accelerated motion in the opposite direction.

The path covered during uniformly decelerated motion of the point is

$$S_1 = v_0 t - \frac{|a_t|\, t^2}{2} = 3 \times 3 - \frac{1 \times 3^2}{2} = 4.5\text{ m}.$$

The path traversed during uniformly accelerated motion of the point is

$$S_2 = v_0 t + \frac{|a_t|\, t^2}{2} = 0 + \frac{1 \times 7^2}{2} = 24.5\text{ m}.$$

The path covered by the point in 10 seconds is

$$S = S_1 + S_2 = 4.5 + 24.5 = 29 \text{ m}.$$

The distance of the point from the origin at the end of the tenth second is found from formula (87), assuming $a_t = -1$:

$$s = s_0 + v_0 t + \frac{a_t t^2}{2} = 5 + 3 \times 10 - \frac{1 \times 10^2}{2} = -15 \text{ m}.$$

At the end of the tenth second the point was located 15 m from the origin in the direction opposite the positive axis for reckoning distance. The distance s can readily be found in another way. It is obvious that

$$s = s_0 + S_1 - S_2 = 5 + 4.5 - 24.5 = -15 \text{ m}.$$

Problem 67. A train is moving along a curve in uniformly accelerated motion and in 3 minutes attains a speed of $v = 54$ km/h after leaving the station. Determine the acceleration of the train at the expiration of 2 minutes after leaving the station if the radius of the curve of the railway track is $\rho = 500$ m.

Solution. Setting $v = 54 \text{ km/h} = \frac{54\,000}{3600} = 15$ m/s, $v_0 = 0$ and $t = 3 \text{ min} = 180$ s, we find [from formula (85)] the tangential acceleration of the train to be

$$a_t = \frac{v}{t} = \frac{15}{180} = \frac{1}{12} \text{ m/s}^2.$$

Since the train is in uniformly accelerated motion, its tangential acceleration is constant.

The normal acceleration of the train given by (72) is $a_n = v^2/\rho$ and depends on the velocity, that is, differs at different times.

We find the velocity of the train at the end of the second minute after leaving the station:

$$v = v_0 + v_t t = 0 + \frac{1}{12} \times 120 = 10 \text{ m/s}.$$

Then the normal acceleration of the train at that time is

$$a_n = \frac{v^2}{\rho} = \frac{10^2}{500} = 0.2 \text{ m/s}^2.$$

The total acceleration of the train at the end of the second minute is

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{1}{5}\right)^2} \approx 0.217 \text{ m/s}^2.$$

Problem 68. A train moving with a velocity of 36 km/h puts on the brakes at a distance of 200 metres from the stop. Assuming the train to be in uniformly variable motion, find the braking time and the tangential acceleration acquired by the train during braking.

Solution. To find the unknown time t of braking and the tangential acceleration a_t of the train we have a system of two equations, (85) and (86):

$$v = v_0 + a_t t \quad \text{and} \quad S = v_0 t + \frac{a_t t^2}{2}.$$

The initial speed of the train $v_0 = 36$ km/h $= 10$ m/s. The final speed of the train is $v = v_0 + a_t t = 0$, whence $a_t = -v_0/t$. Substituting this expression of a_t into the formula for the path, we have

$$S = v_0 t - \frac{v_0 t}{2} = \frac{v_0 t}{2}.$$

From this we get the braking time:

$$t = \frac{2S}{v_0} = \frac{2 \times 200}{10} = 40\ \text{s}.$$

The tangential acceleration of the train is

$$a_t = -\frac{v_0}{t} = -\frac{10}{40} = -0.25\ \text{m/s}^2.$$

The minus sign shows that the acceleration is in the opposite direction to that of the motion of the point, that is, the train is in decelerated motion.

Problem 69. A stone drops into a well, and 4 seconds later a splash of water is heard. Find the depth of the well: (1) assuming the sound to be propagated instantaneously, (2) taking into account that the speed of sound is $v_{\text{sound}} = 333$ m/s.

Solution. (1) Ignoring the time of propagation of sound, we determine the depth of the well as: $h = v_0 t + \frac{gt^2}{2} = 0 + \frac{9.81 \times 4^2}{2} \approx$ ≈ 78.5 m.

(2) The time t_1 of fall of the stone to the depth h of the well is found from the formula $h = gt_1^2/2$, whence $t_1 = \sqrt{2h/g}$.

Sound covers the distance h in a time

$$t_2 = \frac{h}{v_{\text{sound}}}.$$

Obviously the time interval t from the start of fall of the stone up to the time when the splash was heard is

$$t = t_1 + t_2 = \sqrt{\frac{2h}{g}} + \frac{h}{v_{\text{sound}}}.$$

Substituting into this equation $t = 4$ s, $g = 9.81$ m/s^2 and $v_{\text{sound}} =$ $= 333$ m/s, we obtain $h \approx 70.4$ m.

Problem 70. The rectilinear uniformly variable motion of a point is given by the equation

$$s = 40 + 2t + 0.5t^2,$$

where s is in metres and t is in seconds. Construct the graph of motion, velocity, and acceleration of the point over the time interval $t = 0$ to $t = 10$ s and find (analytically and graphically) the velocity v, the path covered S, and the acceleration a of the point after a lapse of 10 seconds.

Solution. To construct the graph of motion of the point, we set up a table, give various values to the time t and compute from the given equation the corresponding values of the distance s.

t, s	0	1	2	3	4	5	6	7	8	9	10
s, m	40	42.5	46	50.5	56	62.5	70	78.5	88	98.5	110

Assuming the scales of distance $\mu_s = 2$ m/mm and time $\mu_t = 0.2$ s/mm, we construct (Fig. 162) a graph of the motion of the point,

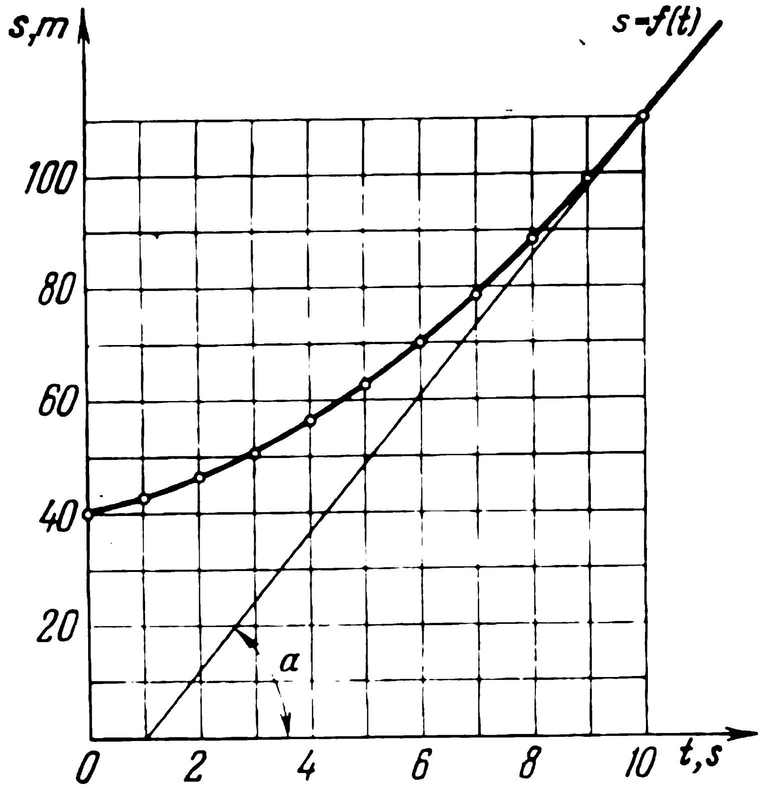


FIG. 162

that is, the curve $s = f(t)$. This curve is a parabola with vertex not located at the coordinate origin. Since the graph of any function of the form $y = ax^2 + bx + c$ is a parabola, it follows that the graph of uniformly variable motion of a point is always a parabola. To construct the graph of velocity, we determine its algebraic value using formula (64):

$$v=\frac{ds}{dt}=2+0.5\times 2t=2+t.$$

Since the relationship $v=f'(t)$ is given by an equation of the first degree, the graph of the velocity is a straight line, and to construct it we have to know only two points through which it passes; for example: for $t=0$, $v=2$ m/s, for $t=10$ s, $v=2+10=12$ m/s.

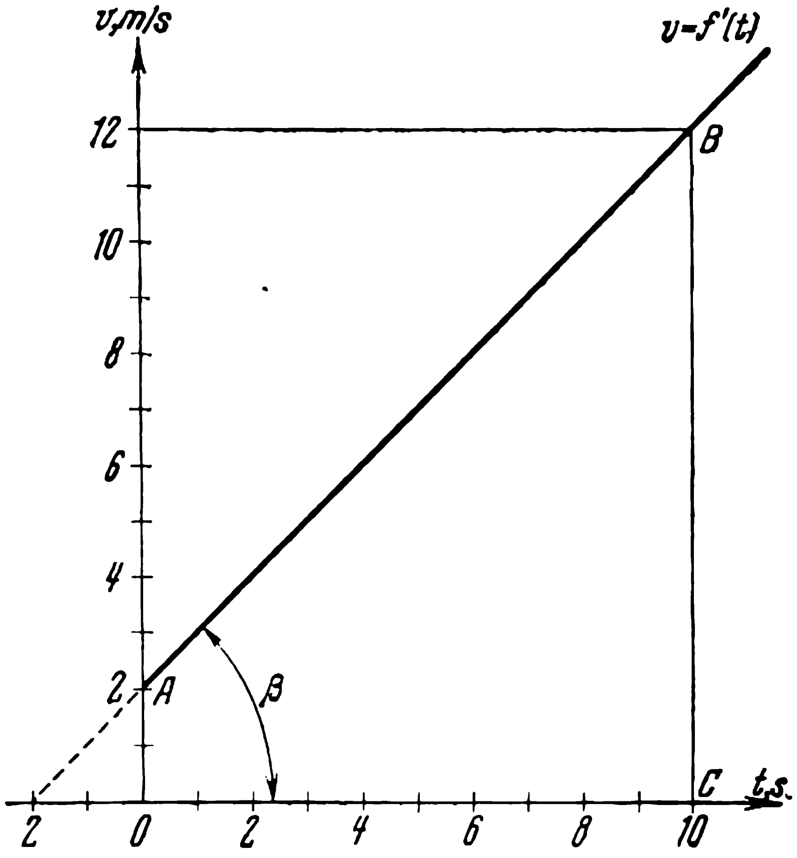


FIG. 163

We take the scales of velocity $\mu_v=0.2$ m/(s·mm) and time $\mu_t=0.2$ s/mm, and on the basis of the coordinates t and v of two points thus found we construct (Fig. 163) a straight line, which is the graph of velocity of the given uniformly variable motion of the point.

The velocity of any uniformly variable motion of a point is given by the formula $v=v_0+a_t t$, which is an equation of the first degree in t and v; and therefore the graph of velocity of uniformly variable motion of a point is always a certain straight line.

The velocity of the point at any time can also be found graphically on the basis of the graph of its motion. In order to find the velocity of the point at $t=10$ s by this means, we draw a tangent line to the corresponding point of the graph (Fig. 162) and then find the tangent of the angle between this tangent line and the positive direction of the t-axis. From Fig. 162 we find $\tan\alpha\approx 11/9$. Then by formula (82),

$$v=\frac{\mu_s}{\mu_t}\tan\alpha\approx\frac{11}{9}\times\frac{2\text{ m/mm}}{0.2\text{ s/mm}}\approx 12\text{ m/s}.$$

The acceleration of a point in rectilinear motion is always $a = a_t = dv/dt$. Thus in our case $a = \frac{d}{dt}(2 + t) = 1$ m/s^2 = constant. Let us now find the acceleration of the point graphically. Since the graph of the velocity of uniformly variable motion is a straight line, the tangent to this graph at any point coincides with the given straight line, and therefore it follows from the general rule for graphical determination of tangential acceleration (see Sec. 6.2) that the algebraic value a_t of the tangential acceleration of uniformly variable motion of a point is equal to the tangent of the angle β between the time axis and the rectilinear graph of velocity of that motion multiplied by the velocity scale μ_v and divided by the time scale μ_t that are assumed in the construction of the velocity graph.

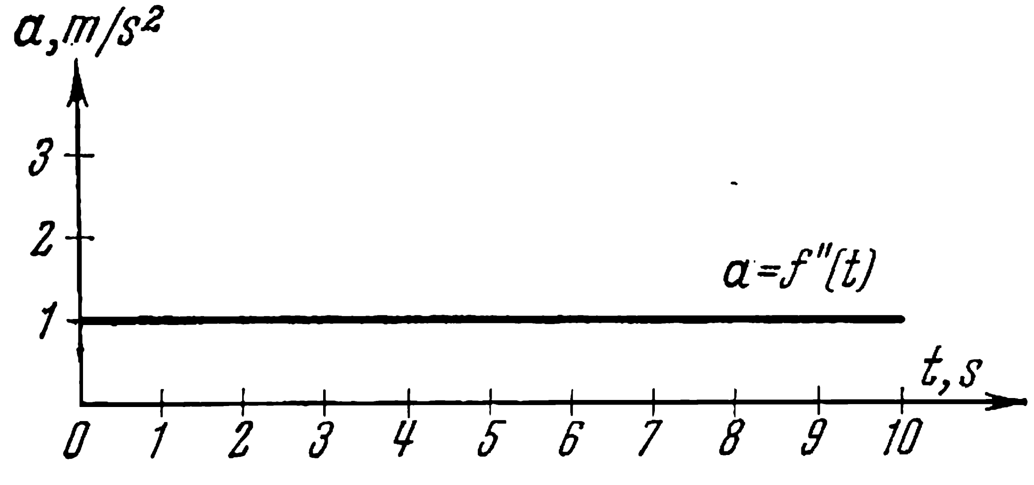


FIG. 164

In our case, $\tan\beta = 12/12 = 1$. Hence, the acceleration of the point is

$$a = a_t = \frac{\mu_v}{\mu_t}\tan\beta = 1 \times \frac{0.2\text{ m/(s}\cdot\text{mm)}}{0.2\text{ s/mm}} \times 1 = 1\text{ m/s}^2.$$

Since the acceleration of uniformly variable motion is a constant quantity, the graph of tangential acceleration of uniformly variable motion is always a straight line parallel to the time axis. The graph of acceleration, $a = f''(t)$, for the given case of motion of the point is depicted in Fig. 164.

It now remains to find the path S covered by the point in 10 seconds. From the equation of motion of the point, $s = f(t)$, we find the distance of the point from the reference origin at the initial time and at $t = 10$ s. At $t = 0$, $s = s_0 = 40$ m, at $t = 10$ s, $s = 40 + 2 \times 10 + 0.5 \times 10^2 = 110$ m. Then the path covered by the point is $S = s - s_0 = 110 - 40 = 70$ m.

Graphically, this path is found either from the graph of the distance or from the velocity graph. To determine the path via the distance graph, we measure (on the graph of Fig. 162) the ordinate that corresponds to $t = 10$ s, and from it subtract the initial ordinate. The resulting line segment

$$55 \text{ mm} - 20 \text{ mm} = 35 \text{ mm}$$

is then multiplied by the distance scale and the result is

$$35 \text{ mm} \times 2 \text{ m/mm} = 70 \text{ m}.$$

To find the path covered from the velocity graph (Fig. 163), measure the area of *OABC* equal to

$$\frac{OA+BC}{2} \times OC = \frac{10+60}{2} \times 50 = 1750 \text{ mm}^2.$$

From formula (84),

$$S = \mu_v \mu_t \times \text{area } OABC = 0.2 \frac{\text{m}}{\text{s}\cdot\text{mm}} \times 0.2 \frac{\text{s}}{\text{mm}} \times 1750 \text{ mm}^2 = 70 \text{ m}.$$

CHAPTER XIV

ELEMENTARY TYPES OF MOTION OF A RIGID BODY

Sec. 64. Translational motion

Translational motion of a rigid body is motion in which every straight line fixed in the body remains parallel, during motion, to its original position.

Instances of translational motion are: the motion of the body of an automobile moving in a straight line, the motion of a piston of an engine. It is not correct however to think that in translational motion of a body the trajectories of its points must necessarily be straight lines. For example, the connecting rod AB (Fig. 165) connecting the crankshafts O_1A and O_2B of two adjacent wheels of a railway locomotive performs stranslational motion, although its points move in circles. Indeed, when the crankshafts O_1A and O_2B turn on their axes O_1 and O_2, the position of the rod AB changes. Given the equal lengths of the crankshafts and given the length of the connecting rod as equal to the distance between the axes O_1O_2, the quadrilateral O_1ABO_2 will always remain a parallelogram and hence the rod AB will always be parallel to the base O_1O_2, that is to say, it is in motion and remains parallel to its initial position. At the same time, the points A and B of the rod and, hence, all its other points move in circles, whose radii are equal to the length of the crankshaft.

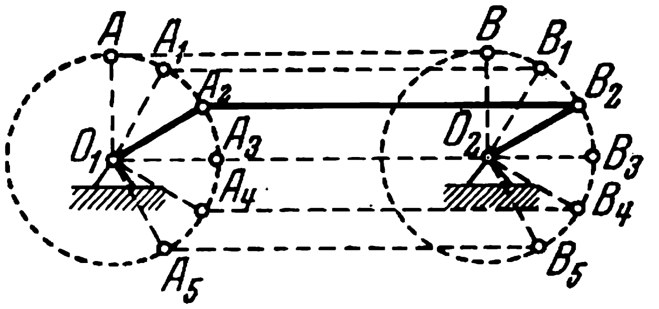


FIG. 165

When a rigid body is in translational motion, the trajectories of the points of the body are capable of describing any kinds of curves: rectilinear motion of the body is only a special case of translational motion.

Also note that the term translational motion is applicable only to the motion of a body and not to that of a single point. The concept of "moving and remaining parallel to its initial position" is in no way applicable to a point (particle) without dimensions.

Theorem. *When a rigid body is in translational motion, all its points move along the same trajectories (they coincide when superimposed) and at every instant of time have the same velocities and the same accelerations.*

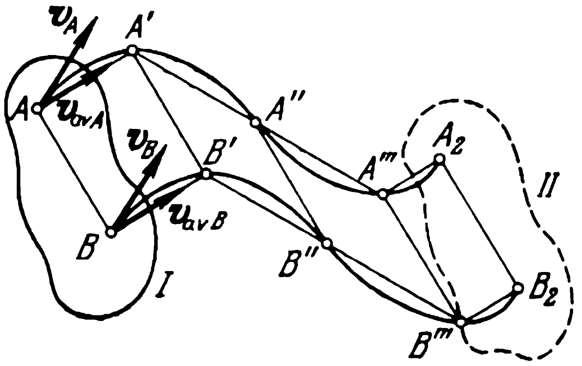


FIG. 166

Proof. Suppose a body moving translationally is displaced during a time interval t from position I to position II (see Fig. 166). We partition the given time interval into a large number n of small subintervals Δt and note the position of an arbitrary straight line AB (fixed in the body) at the end of each subinterval.

The line segments AB, $A'B'$, $A''B''$, ..., A_2B_2 constitute the sequence of positions of one and the same line AB moving together with the rigid body. Consequently, $AB = A'B' = A''B'' = \dots = A_2B_2$. Besides, on the basis of the definition of translational motion, these lines are all parallel among themselves:

$$AB \parallel A'B', \quad A'B' \parallel A''B'' \quad \text{and so forth.}$$

Therefore, if we join the points A, A', A'', ..., A_2 and the points B, B', B'', ..., B_2 with straight-line segments, then the lines AA', $A'A''$, . . ., $A'''A_2$ will be respectively equal and parallel to the lines BB', $B'B''$, . . ., $B'''B_2$ since they are segments lying between equal and parallel lines. From this it follows that the polygonal lines $AA'A'' \dots A_2$ and $BB'B'' \dots B_2$ connecting the different positions of the points A and B have equal and parallel sides and for this reason may be brought to coincidence with each other. This holds true for any number n of intervals, that is for any number of marked positions of the points A and B. In the limit, as $n \to \infty$ and $\Delta t \to 0$, the polygonal lines $AA'A'' \ldots A_2$ and $BB'B'' \ldots B_2$ become the trajectories of the points A and B of the body. These trajectories can also obviously be brought to coincidence with each other and therefore constitute identical curves.

Now we will prove that the points A and B have at every instant of time equal velocities and equal accelerations.

The average velocity of A when passing to position A' during the subinterval of time Δt is the vector $\mathbf{v}_{\text{av}\,A} = \overrightarrow{AA'}/\Delta t$. This vector (see Sec. 55) has the direction of the vector $\overrightarrow{AA'}$ (the displacement vector of the point) and is equal in magnitude to the ratio of the length of line segment AA' to the time interval Δt.

The average velocity of point B as it moves to position B' during the same subinterval of time Δt is the vector $\mathbf{v}_{\text{av}\,B} = \overrightarrow{BB'}/\Delta t$, which has the direction of the vector $\overrightarrow{BB'}$ and is equal in magnitude to the ratio of the length of the segment BB' to the same time interval Δt.

However, as has been proved above, the line segments $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ are equal and parallel. Hence, $\mathbf{v}_{\text{av}\,A} = \mathbf{v}_{\text{av}\,B}$.

From this, passing to the limit as $\Delta t \to 0$, we have

$$\lim_{\Delta t \to 0} \mathbf{v}_{\text{av}\,A} = \lim_{\Delta t \to 0} \mathbf{v}_{\text{av}\,B}.$$

And since the velocity of the point at each instant of time is equal to the limit of its average velocity, we obtain

$$\mathbf{v}_A = \mathbf{v}_B.$$

Since the velocity vector $\mathbf{v}_A$ of point A is equal to the velocity vector $\mathbf{v}_B$ of point B at each instant, it is obvious that the vector $\Delta\mathbf{v}_A$ (the increment in the velocity) of point A during some time interval Δt is equal to the vector $\Delta\mathbf{v}_B$ (the increment in the velocity) of point B during that interval of time:

$$\Delta\mathbf{v}_A = \Delta\mathbf{v}_B.$$

From this it follows that the vectors of the average accelerations (over a given time interval Δt) of the points in question are equal as well:

$$\frac{\Delta \mathbf{v}_A}{\Delta t} = \frac{\Delta \mathbf{v}_B}{\Delta t}.$$

Passing to the limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}_A}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}_B}{\Delta t}.$$

But each part of the last equality expresses [formula (69)] the acceleration of the corresponding point at the given time. Hence

$$\mathbf{a}_A = \mathbf{a}_B.$$

We have proved that the points A and B move in the same way. These points of the body were chosen at random and so the conclusions that we have drawn may be extended to all other points of a body in translational motion.

From the theorem just proved it follows that the *translational motion of a body is fully determined by the motion of any one of its points.* Thus, the problem of studying the translational motion of a rigid body reduces to the already considered problems of the kinematics of a point.

The velocity and acceleration of all points of a body in translational motion are the same and termed the velocity and acceleration of that body.

Note that only in the case of translational motion can we speak of the velocity and acceleration of a body. In all other cases, different points of the body have distinct velocities and distinct accelerations.

Sec. 65. Rotational motion

*Rotational motion is the motion of a rigid body in which all its points lying on a certain straight line, called the axis of rotation, remain fixed.**

---

* A rotating body need not necessarily have its points on the geometric axis of rotation, that is, it may not have fixed points. Examples are: a wheel mounted on an axle, or a human being sitting on merry-go-round. But if the points situated on the geometric axis of rotation are fixed in (attached to) a rotating body, then they will remain fixed when the body is in rotation.

To achieve rotational motion, it is sufficient to fix any two points of a body (for example, by means of bearing A and bearing B in Fig. 167); then the straight line passing through these two points will be the axis of rotation of the body.

In rotational motion, the different points of the body will, generally speaking, move in different ways. But here too we can find kinematic characteristics that will be common to all points of the body.

Suppose a rigid body (depicted, for the sake of simplicity, in the form of a cylinder in Fig. 167) is in rotation about a fixed axis z. Draw through the axis of rotation z a fixed half-plane P and a half-plane Q fixed in the rotating body. *The angle* φ *between two half-planes passing through the axis of rotation of the body* (*the fixed half-plane and the plane rigidly attached to the rotating body*) *is called the angle of rotation of the body.*

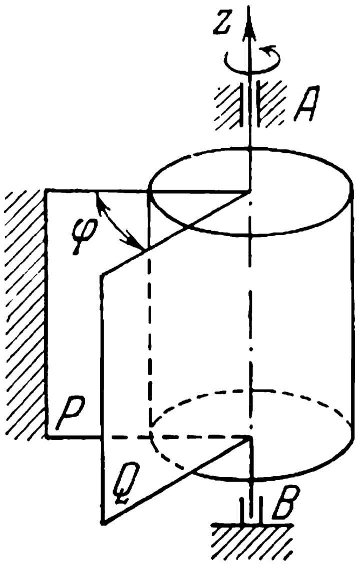


FIG. 167

On the z-axis (the axis of rotation) we establish a positive direction and agree to regard the angle of rotation positive when it is reckoned from the fixed plane P in the counterclockwise direction if one looks at the angle from the positive end of the axis of rotation. By specifying the magnitude and sign of the angle of rotation φ, we completely determine the position of the half-plane Q and the rotating body to which it is rigidly attached.

As the body rotates about the z-axis, the angle of rotation of the body changes with time, which means it is some function of the time:

$$\varphi = f(t). \tag{91}$$

The equation (91) *establishes a relationship between the angle of rotation of a body and the time of its motion and is called the equation of rotational motion of a body.*

This equation fully determines the rotational motion of a body because if we know the concrete relationship $\varphi = f(t)$, we can always (at any instant of time t) find the corresponding value of the angle of rotation, φ, of the body and thus determine its position at that time.

In mechanics, the angle of rotation is usually measured in nondimensional units, that is, in radians. At times, in practical problems, the angle of rotation is expressed by the number of revolutions of the body, φ_{rev}. Taking into account that one revolution (a rotation through 360°) of a body corresponds to 2π radians, we have a relationship between the angle of rotation φ of the body in radians and the number of revolutions of the body, φ_{rev}:

$$\varphi = 2\pi\varphi_{\text{rev}}. \tag{92}$$

Suppose at time t the position of a body is given by the angle of rotation φ and at time $t+\Delta t$, by the angle of rotation $\varphi+\Delta\varphi$.

The ratio of the increment $\Delta\varphi$ in the angle of rotation of a body over a time interval Δt to the value of that time interval is termed the average angular velocity of the body during the time interval Δt. Denoting the average angular velocity by ω_{av}, we have

$$\omega_{\text{av}} = \frac{\Delta\varphi}{\Delta t}. \tag{93}$$

The average angular velocity is, generally, dependent on the magnitude of the time interval and does not give any idea of the rate of rotation of the body at the given time. But the smaller the time interval Δt we take (starting at a given time t), the more precisely will the average angular velocity characterize the rate of rotation at a given time. Therefore, *the true angular velocity or the angular velocity at a given time is the limit to which the average angular velocity of a body tends during the time interval Δt, starting at the given time t, as $\Delta t \to 0$.*

Denoting the angular velocity of a body at a given instant of time by the letter ω, we have

$$\omega = \lim_{\Delta t\to 0} \omega_{\text{av}} = \lim_{\Delta t\to 0} \frac{\Delta\varphi}{\Delta t}.$$

Since the angle of rotation, φ, of a body is a function of the time, $\varphi = f(t)$, it follows that $\lim\limits_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t}$ is the derivative of that function.

Thus we find that the *angular velocity of a body at a given time is equal to the derivative of the angle of rotation of the body with respect to time*:

$$\omega = \frac{d\varphi}{dt}. \tag{94}$$

The value of the angular velocity of a body may, at a given time, be positive or negative, depending on the sense of rotation of the body. When a body is in counterclockwise rotation, if one looks from the positive end of the axis of rotation, then $\Delta\varphi > 0$, $d\varphi/dt > 0$ and the angular velocity ω is positive. If a body is in clockwise rotation, the angular velocity is negative. Thus the sign of angular velocity indicates in which direction the body is rotating at a given time.

Using the definition of angular velocity, we can find its dimensions as well:

$$[\omega] = \lim_{\Delta t \to 0} \frac{[\Delta\varphi]}{[\Delta t]} = \frac{[\text{angle}]}{[\text{time}]}.$$

Since angles in mechanics are measured in radians and time is measured in seconds, the *angular velocity is measured in radians per second.**

The unit of angular velocity is then given the designation: rad/s.

In practical cases, the angular velocity of a body is not usually given in radians per second but by what is called the frequency of rotation. This is expressed by the number of revolutions per minute and is symbolized by n. It is easy to find the relationship between ω and n.

Since one revolution of a body corresponds to its rotation through an angle of 2π radians, we have

$$\omega = \frac{2\pi n \text{ rad}}{60 \text{ s}} = \frac{\pi n}{30} \text{ rad/s}.$$

---

* By measuring angles in radians, that is, by the ratio of the arc lying between two sides of an angle to the radius of the arc, we express angles by means of nondimensional numbers.

We thus obtain an important practical relationship between the angular velocity of a body ω in radians per second and its angular velocity n in revolutions per minute:

$$\omega = \frac{\pi n}{30}. \tag{95}$$

Bear in mind that in this formula, ω is always expressed in rad/s and n in rpm (revolutions per minute).

In approximate calculations, we take $\pi/30 \approx 0.1$; then $\omega \approx 0.1n$.

If a body is in nonuniform rotation, that is, if the increments in the angle of rotation, φ, over equal time intervals are not equal, then its angular velocity ω varies with time and is thus some function of the time:

$$\omega = f'(t).$$

The quantity that describes the rate of change of angular velocity of a body is termed its angular acceleration.

Suppose at time t a body has angular velocity ω and at time $t + \Delta t$, angular velocity $\omega + \Delta\omega$.

The ratio of the increment $\Delta\omega$ in angular velocity of a body during a time interval Δt to that interval of time is termed the average angular acceleration of the body over the time interval Δt.

Denoting the average angular acceleration of a body by ε_{av} (ε is the Greek letter epsilon), we have

$$\varepsilon_{av} = \frac{\Delta\omega}{\Delta t}. \tag{96}$$

As $\Delta t \to 0$, the average angular acceleration approaches a limit called the true or merely the angular acceleration of the body at the given instant of time.

The limit to which the average angular acceleration of a body tends during the interval of time Δt, beginning with the given time t, as $\Delta t \to 0$, is called the angular acceleration of the body at the given time.

Denoting the angular acceleration of a body at a given time by the letter ε, we have

$$\varepsilon = \lim_{\Delta t \to 0} \varepsilon_{av} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}.$$

But since the angular velocity ω of a body is a function of time, $\omega = f'(t)$, it follows that the limit $\lim\limits_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}$ is a derivative of that function. Hence,

$$\varepsilon=\frac{d\omega}{dt}=\frac{d}{dt}\frac{d\varphi}{dt}=\frac{d^2\varphi}{dt^2}=f''(t). \tag{97}$$

The angular acceleration of a body at a given time is equal to the first derivative of the angular velocity of the body with respect to time, or the second derivative of the angle of rotation of the body with respect to time.

In the case of accelerated rotation of a body, the increment $\Delta\omega$ in its angular velocity and, hence, the angular acceleration ε are positive; in the case of decelerated rotation, it is negative. From this it follows that *if the sign of angular acceleration of a body coincides with the sign of its angular velocity, then the body is in accelerated rotation, if their signs differ, the body is in decelerated rotation.*

The dimensions of angular acceleration are:

$$[\varepsilon]=\lim_{\Delta t\to 0}\frac{[\Delta\omega]}{[\Delta t]}=\frac{[\text{angular velocity}]}{[\text{time}]}.$$

Since angular velocity is measured in radians per second, angular acceleration is measured in radians per second per second.

The unit of angular acceleration is rad/s^2.

Problem 71. The rotation of a disc about a fixed axis is given by the equation $\varphi = 180t - 15t^2$ (φ in radians and t in seconds). Find the angular velocity ω and the angular acceleration ε of the disc at times: $t = 0$, $t = 6$ s, and $t = 7$ s.

Solution. We find the angular velocity of the disc from formula (94):

$$\omega=\frac{d\varphi}{dt}=\frac{d}{dt}(180t-15t^2)=(180-30t)\ \text{rad/s};$$

for $t = 0$, $\omega = 180$ rad/s; for $t = 6$ s, $\omega = 180 - 30 \times 6 = 0$; for $t = 7$ s, $\omega = 180 - 30 \times 7 = -30$ rad/s.

The angular acceleration of the disc is found from formula (97):

$$\varepsilon=\frac{d\omega}{dt}=\frac{d^2\varphi}{dt^2}=\frac{d}{dt}(180-30t)=-30\ \text{rad/s}^2=\text{constant}.$$

At the initial time ($t = 0$), the angular velocity and the angular acceleration of the disc have different signs, which means the disc is in decelerated rotation. At $t = 6$ s, the angular velocity of the disc vanishes, after which the disc begins to pick up speed (the angular velocity and angular acceleration have the same sign).

Sec. 66. The trajectories, velocities and accelerations of points of a rotating rigid body

When a body is in rotation about a fixed axis, all the points of the body describe circles lying in planes perpendicular to the axis of rotation, the z-axis (see Fig. 168). *The centres of these circles lie on the axis of rotation and, hence, the radius of each of them is equal to the distance of the appropriate point of the body from the axis of rotation.*

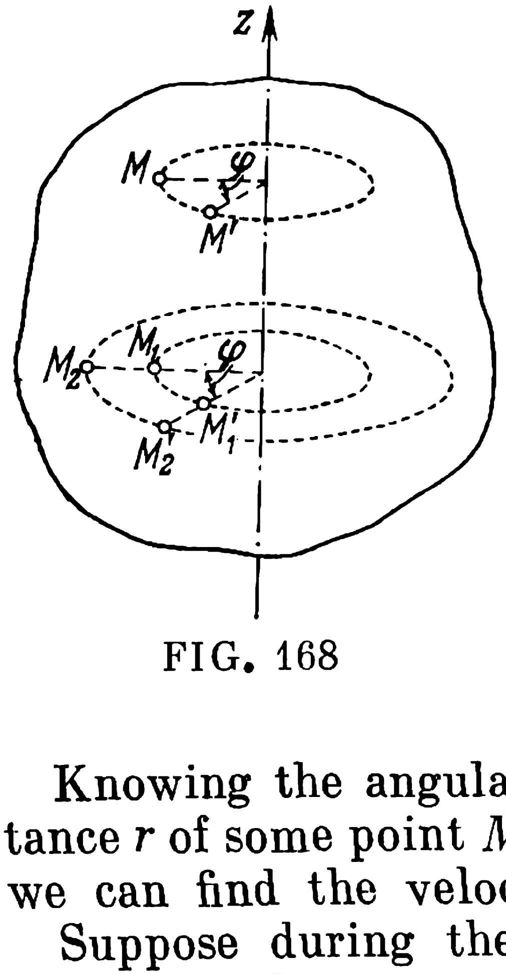


FIG. 168

Obviously, the radii of all given circles turn, during the same time interval Δt, through the same angle $\Delta\varphi$, which is equal to the increment in the angle φ of rotation of the body during that time interval; however, points lying at different distances from the axis of rotation (points M_1 and M_2 in Fig. 168) will describe arcs of different length.

Knowing the angular velocity ω of the body and the distance r of some point M of the body from the axis of rotation, we can find the velocity* v of that point.

Suppose during the interval of time Δt, which corresponds to the increment $\Delta\varphi$ in the angle of rotation of the body, a given point moves from position M to M' (Fig. 168). The arc length MM' covered by point M in its trajectory is equal in absolute value to the increment Δs in the distance s of point M from that point of the given trajectory that is taken as the origin of the coordinate system.

---

* The velocity of a point of a rotating body that characterizes (by its absolute value) the rate of motion along the arc of the trajectory (measured in linear units) is sometimes called the *linear velocity* in contrast to the angular velocity of the body that describes the rate of change of the angle of its rotation.

Now, as we know, the numerical value of the velocity of a point is equal to the derivative, with respect to time, of the distance s of the given point from the origin:

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

But the arc length of a circle is equal to its radius multiplied by the appropriate centre angle in radians, that is, in the given case we have $\Delta s = r\,\Delta\varphi$.

Substituting the value of Δs into the preceding equation, we get

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\,\Delta\varphi}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = r\frac{d\varphi}{dt} = r\omega.$$

The numerical value of the velocity of a point of a rotating body is equal to the product of the angular velocity of the body into the distance of the given point from the axis of rotation:

$$v = r\omega. \tag{98}$$

The velocity vector $\mathbf{v}$ of the point is directed along the tangent to the trajectory of the point and, hence, is perpendicular to its radius of rotation and is in the direction of motion of the point. From formula (98) it follows that the *velocities of points of a rotating body are proportional to the distances of the points from the axis of rotation.*

Since, in the formula $\Delta s = r\,\Delta\varphi$, the increment in the angle $\Delta\varphi$ must necessarily be expressed in radians, it follows that the angular velocity ω in (98) must necessarily be expressed in rad/s, rad/min and so on, but not in revolutions per second or in rpm. Only in that case will the proper dimensions of the velocity v result. In practical situations, it is often required to find the velocities of points lying on the lateral surface of a rotating cylindrical body (shaft, pulley, etc.) which is sometimes then called the *peripheral velocity*. If the angular velocity of the body is expressed in rpm, then the peripheral velocity

$$v = r\omega = \frac{d}{2} \times \frac{\pi n}{30} = \frac{\pi n d}{60}, \tag{99}$$

where d is the diameter of the rotating cylindrical body and n is the number of revolutions of the body per minute.

In any rotational motion of a body, the velocity $\mathbf{v}$ of its points unavoidably varies (only in direction in the case of uniform rotational motion or in direction and magnitude in nonuniform rotational motion); consequently, the points of the rotating body always move with a certain acceleration.

The acceleration of a point of a rotating body, like the acceleration of any curvilinear motion, may be decomposed into tangential acceleration $\mathbf{a}_t$, which in this case is frequently called *rotary* acceleration, and normal acceleration $\mathbf{a}_n$, which in this case is called *centripetal* acceleration.

If for the tangential and normal acceleration of a point in formulas (70) and (72) we put $v = r\omega$ and the radius of curvature ρ of the trajectory is made equal to the radius of the circle described by the point, that is, the distance r of the point from the axis of rotation, we then have

$$a_t = \frac{dv}{dt} = \frac{d}{dt} r\omega = r\frac{d\omega}{dt} = r\varepsilon,$$

$$a_n = \frac{v^2}{\rho} = \frac{r^2\omega^2}{r} = r\omega^2.$$

The numerical value of rotary acceleration of a point (that is, the tangential acceleration of the point of a rigid body rotating about a fixed axis) *is equal to the product of the angular acceleration of the body by the distance of the given point from the axis of rotation*:

$$a_t = r\varepsilon. \tag{100}$$

The numerical value of the centripetal acceleration of a point (that is, the normal acceleration of the point of a rigid body rotating about a fixed axis) *is equal to the square of the angular velocity of the body multiplied by the distance of the given point from the axis of rotation*:

$$a_n = r\omega^2. \tag{101}$$

The direction of the vector of rotary acceleration of a point coincides with the direction of the vector of its velocity in the case of accelerated rotation of the body

(Fig. 169*a*) and is opposite the velocity in the case of decelerated rotation (Fig. 169*b*).

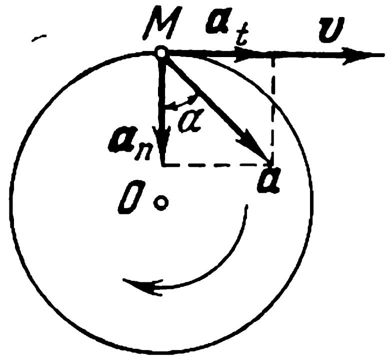
Now, the vector of centripetal acceleration of a point is always directed along the radius of the circle described by the point towards the centre of the circle (whence the name).

If we know the rotary and centripetal components of the acceleration **a** of a point, it is always possible (if necessary) to find the magnitude and direction of that acceleration, which is a diagonal of a rectangle constructed on the vectors $\mathbf{a}_t$ and $\mathbf{a}_n$ (Fig. 169).

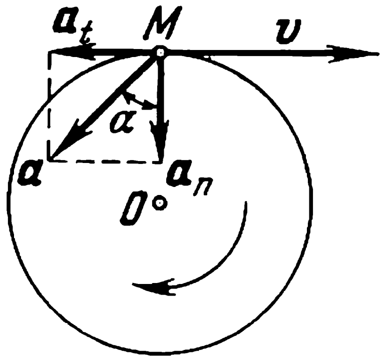
The absolute value of this acceleration is

$$a = \sqrt{a_t^2 + a_n^2}$$

$$= \sqrt{r^2\varepsilon^2 + r^2\omega^4} = r\sqrt{\varepsilon^2 + \omega^4}.$$



(*a*) *Accelerated rotation*



(*b*) *Decelerated rotation*

FIG. 169

The acute angle α between the direction of the acceleration vector **a** of the point and the direction of the radius (or normal to the trajectory) is found from the formula

$$\tan\alpha = \frac{a_t}{a_n} = \frac{|\varepsilon|}{|\omega^2|}.$$

In order to obtain the proper dimensions of the accelerations a_t, a_n and a in formulas (100) and (101), substitute ε in radians per second per second and ω in radians per second.

Problem 72. A fan does 4200 rpm. What is the permissible diameter of the fan if the peripheral velocity must not exceed 88 m/s?

Solution. From formula (99), $v = \frac{\pi dn}{60}$, we immediately get the desired answer:

$$d = \frac{60v}{\pi n} \leqslant \frac{60 \times 88}{3.14 \times 4200} \quad \text{or} \quad d \leqslant 0.4 \text{ m}.$$

Problem 73. A pulley of diameter 600 mm turns on its axis. The velocity of the points on its rim is equal to 1.5 m/s at a given time. Find the angular velocity of the pulley in rpm.

Solution. From formula (98), $v = r\omega$, we compute first the angular velocity ω of the pulley in radians per second:

$$\omega = \frac{v}{r} = \frac{1.5}{0.3} = 5 \text{ rad/s}.$$

Using the relation (95), $\omega = \frac{\pi n}{30}$, we find the angular velocity n of the pulley in rpm:

$$n = \frac{30v}{\pi} = \frac{30 \times 5}{3.14} \approx 47.8 \text{ rpm}.$$

Problem 74. Referring to Fig. 170, at a given time a flywheel has an angular velocity $\omega = 2\pi$ rad/s and an angular acceleration $\varepsilon = -3$ rad/s². Find the velocity, the rotary, centripetal and total acceleration of point M located on the flywheel at a distance of 0.8 m from the axis of rotation.

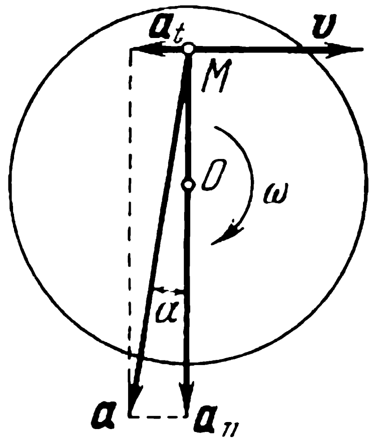


FIG. 170

Solution. The velocity of the point $v = r\omega = 0.8 \times 2\pi = 1.6\pi \approx 5$ m/s. The rotary acceleration of the point is $a_t = r\varepsilon = 0.8 \times 3 = 2.4$ m/s². Since ω and ε have opposite signs, the flywheel is in decelerated rotation, and so the acceleration $\mathbf{a}_t$ of the point is in the opposite direction to that of its velocity.

The centripetal acceleration of the point is

$$a_n = r\omega^2 = 0.8 \times (2\pi)^2 = 3.2\pi^2 \approx 31.5 \text{ m/s}^2.$$

The total acceleration of the point is

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{2.4^2 + 31.5^2} \approx 31.6 \text{ m/s}^2,$$

$$\tan\alpha = \frac{a_t}{a_n} = \frac{2.4}{31.5} \approx 0.08.$$

Sec. 67. Special cases of rotational motion of a rigid body

1. Uniform rotation

Uniform rotation of a body is rotational motion of the body with a constant angular velocity.

In this case, $\omega = \frac{d\varphi}{dt} =$ constant, whence follows

$$d\varphi = \omega\, dt$$

and

$$\varphi = \int \omega\, dt = \omega t + C.$$

As always, we find the arbitrary constant of integration from the initial conditions. If at $t = 0$ the angle of rotation of the body is $\varphi = \varphi_0$, then, substituting these values of the variables t and φ into the preceding equation, we find $C = \varphi_0$. Consequently,

$$\varphi = \varphi_0 + \omega t. \qquad (102)$$

Equation (102) is called the *equation of uniform rotation of a body*. It permits finding the position of a uniformly rotating body at any instant of time. The difference $\varphi - \varphi_0$ is clearly the angle of rotation of the body during time t.

From the formula (102) it follows that

$$\varphi - \varphi_0 = \omega t.$$

Thus, in the case of uniform rotation of a body, the increment in the angle of rotation of the body during a certain time is equal to the product of its angular velocity into that time.

From this we have

$$\omega = \frac{\varphi - \varphi_0}{t}. \qquad (103)$$

The angular velocity of uniform rotation of a body about a fixed axis is equal to the ratio of the increment in the angle of rotation of the body during the given time interval to the magnitude of that time interval.

The angular acceleration of a body, $\varepsilon = d\omega/dt$, is clearly zero in the case of uniform rotation. From this it follows that in the case of uniform rotation of a body, the rotary acceleration of any one of its points, $a_{\mathrm{t}} = r\varepsilon$, is always equal to zero. Thus the total acceleration of a point of a uniformly rotating body consists solely of the centripetal acceleration

$$a = a_{\mathrm{n}} = r\omega^2$$

and is of course directed along a radius to the centre of the circle described by the given point.

2. *Uniformly variable rotation*

Uniformly variable (uniformly accelerated and uniformly decelerated) rotation of a body is rotational motion of the body such that in equal arbitrarily taken time intervals the angular velocity of the body changes by the same amount.

Obviously, *in uniformly variable rotation, the angular acceleration of the body*, that is, the quantity that describes the rate of change of its angular velocity, is a *constant*:

$$\varepsilon = \frac{d\omega}{dt} = \text{constant}.$$

From this it follows that $d\omega = \varepsilon\, dt$ and

$$\omega = \int \varepsilon\, dt = \varepsilon t + C_1.$$

Denoting the initial angular velocity of the body (that is, the angular velocity at time $t = 0$) by ω_0 and substituting the initial values of the variables t and ω into the preceding equation, we find $C_1 = \omega_0$. Consequently,

$$\omega = \omega_0 + \varepsilon t. \tag{104}$$

From this equation, which is called the *formula of the angular velocity of uniformly variable rotation*, we find

$$\varepsilon = \frac{\omega - \omega_0}{t}.$$

The angular acceleration of uniformly variable rotation is equal to the ratio of the increment in the angular velocity of a body during a given time interval to the magnitude of that time interval. Replacing ω by its value $d\varphi/dt$ in (104), we have

$$\frac{d\varphi}{dt}=\omega_0+\varepsilon t$$

and

$$d\varphi = \omega_0\,dt + \varepsilon t\,dt.$$

Integrating this equation with respect to t, we obtain

$$\varphi=\int \omega_0\,dt+\int \varepsilon t\,dt=\omega_0 t+\frac{\varepsilon t^2}{2}+C_2.$$

If at $t=0$ the angle of rotation of the body $\varphi=\varphi_0$, then by substituting these values of the variables t and φ into the preceding equation, we find the value of the arbitrary constant: $C_2=\varphi_0$. Hence

$$\varphi=\varphi_0+\omega_0 t+\frac{\varepsilon t^2}{2}. \qquad (105)$$

Equation (105) is called the *equation of uniformly variable rotation of a body.*

It is obvious that in the case of uniformly variable rotation of a body, all its points execute a uniformly variable motion along appropriate circles and therefore the formulas of Sec. 63 are applicable to the motion of points of a body in uniformly variable rotation.

Uniformly variable rotation is usually encountered in problems involving the starting and stopping of machines. The angular acceleration of a body, as will be demonstrated in dynamics, is a constant quantity if the rotating moment applied to it is constant.

It is useful to note the analogy that exists between the formulas of the kinematics of a point and the formulas for the rotational motion of a body (see table below). It is easy to see that to pass from one set of formulas to the other requires merely replacing the distance s of the point by the angle of rotation φ of the body, the velocity v of the point by the angular velocity ω of the body, and the tangential acceleration a_t of the point (which characterizes the rate of change of the magnitude of the velocity of the point) by the angular acceleration ε of the body (which describes the rate of change of the magnitude of the angular velocity of the body).

Kinematic characteristics and type of motion		Motion of point		Rotational motion of body	
Equation of motion	General formula		$s=f(t)$		$\varphi=f(t)$
	Uniform motion		$s=s_0+vt$		$\varphi=\varphi_0+\omega t$
	Uniformly variable motion		$s=s_0+v_0t+\frac{a_tt^2}{2}$		$\varphi=\varphi_0+\omega_0t+\frac{\varepsilon t^2}{2}$
Velocity	General formula	linear	$v=\frac{ds}{dt}$	angular	$\omega=\frac{d\varphi}{dt}$
	Uniform motion		$v=\frac{s-s_0}{t}$		$\omega=\frac{\varphi-\varphi_0}{t}$
	Uniformly variable motion		$v=v_0+a_tt$		$\omega=\omega_0+\varepsilon t$
	Dimensions		$\frac{[\text{length}]}{[\text{time}]}$		$\frac{[\text{angle}]}{[\text{time}]}$
Acceleration	General formula	tangential	$a_t=\frac{dv}{dt}=\frac{d^2s}{dt^2}$	angular	$\varepsilon=\frac{d\omega}{dt}=\frac{d^2\varphi}{dt^2}$
	Uniformly variable motion		$a_t=\frac{v-v_0}{t}$		$\varepsilon=\frac{\omega-\omega_0}{t}$
	Dimensions		$\frac{[\text{length}]}{[\text{time}]^2}$		$\frac{[\text{angle}]}{[\text{time}]^2}$

Problem 75. A pulley of radius 0.3 m is rotating uniformly at 1200 rpm. Find its angular velocity ω in radians per second and also the velocity and acceleration of a point lying on the rim of the pulley.

Solution. $\omega = \frac{\pi n}{30} = \frac{3.14 \times 1200}{30} = 125.6$ rad/s, $v = r\omega = 0.3 \times \times 125.6 \approx 3.77$ m/s. Since in the case of uniform rotation of the body the rotary acceleration $a_t = r\varepsilon = 0$, it follows that the total acceleration $a = a_n = r\omega^2 = 0.3 \times 125.6^2 \approx 4733$ m/s^2 and is directed along a radius to the centre of the pulley.

Problem 76. A body initially at rest is set into uniformly accelerated motion and in 10 seconds acquires an angular velocity equal to 30 rad/s. How many revolutions did the body perform during the 10 seconds?

Solution. It is given that the initial angular velocity of the body is $\omega_0 = 0$. Since the rotation of the body is uniformly variable, its angular acceleration $\varepsilon = \frac{\omega - \omega_0}{t} = \frac{30}{10} = 3$ rad/s^2. The increment in the angle of rotation of the body during 10 seconds is equal to $\varphi = = \omega_0 t + \frac{\varepsilon t^2}{2} = \frac{3 \times 10^2}{2} = 150$ rad. Knowing the angle φ of rotation we can use formula (92), $\varphi = 2\pi\varphi_{\text{rev}}$, to find the corresponding number φ_{rev}, the number of revolutions of the body:

$$\varphi_{\text{rev}} = \frac{\varphi}{2\pi} = \frac{150}{2 \times 3.14} \approx 23.9.$$

Problem 77. A flywheel running at 600 rpm was left to itself and due to friction in the bearings stopped after 200 revolutions. Find the angular acceleration of the wheel assuming it to be constant.

Solution. We determine the angle of rotation of the flywheel that corresponds to the 200 revolutions performed prior to stopping:

$$\varphi = 2\pi\varphi_{\text{rev}} = 2\pi \times 200 = 400\pi.$$

The initial angular velocity of the flywheel is

$$\omega_0 = \frac{\pi n}{30} = \frac{\pi \times 600}{30} = 20\pi \text{ rad/s}.$$

The terminal angular velocity of the flywheel is

$$\omega = \omega_0 + \varepsilon t = 0.$$

If the body is in uniformly variable rotation, the angle of rotation is

$$\varphi = \omega_0 t + \frac{\varepsilon t^2}{2}.$$

Solving these two equations simultaneously, we obtain $\varepsilon = -\frac{\omega_0^2}{2\varphi} = = -\frac{(20\pi)^2}{2 \times 400\pi} = -0.5\pi = -1.57$ rad/s^2. The negative value of ε indicates decelerated rotation of the flywheel.

Problem 78. Referring to Fig. 171, on a shaft of radius $r = 10$ cm is wound a rope whose free end is pulled horizontally so that the end moves via the equation $s = 100t^2$ (s is in centimetres, t in seconds), where s is the distance from the fixed vertical line AB. Determine the angular velocity and the angular acceleration of the shaft, and also the total acceleration of a point on the surface of the shaft at time t.

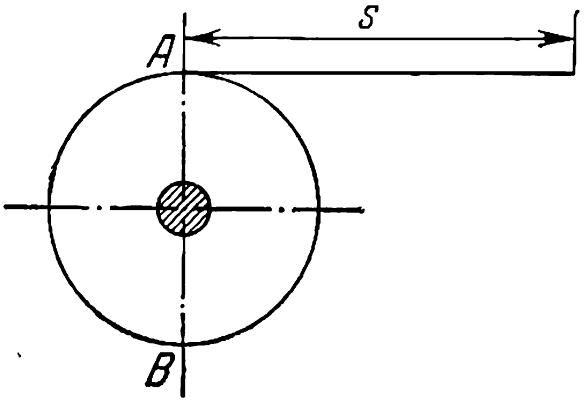


FIG. 171

Solution. The velocity of the rope

$$v = \frac{ds}{dt} = 200t \text{ cm/s}.$$

Since the shaft is turned by the rope being pulled off it, the velocity of the point on the surface of the shaft is equal to that of the rope. Then the angular velocity of the shaft is

$$\omega = \frac{v}{r} = \frac{200t}{10} = 20t \text{ rad/s}.$$

The angular acceleration of the shaft is

$$\varepsilon = \frac{d\omega}{dt} = 20 \text{ rad/s}^2.$$

The total acceleration of the point on the surface of the shaft is given by the formula

$$a = \sqrt{a_t^2 + a_n^2} = r\sqrt{\varepsilon^2 + \omega^4} = 10\sqrt{20^2 + 20^4t^4} = 200\sqrt{1 + 400t^4} \text{ cm/s}^2.$$

Sec. 68. Transmissions of rotational motion

The transmission of rotational motion from one machine to another or within a machine from one shaft to another is performed by a variety of mechanisms called *drives*.

Drives may be subdivided into flexible drives (belt drive, rope drive, chain drive) and direct-contact drives (friction drive, gear drive, etc.).

Shafts with pulleys and wheels mounted on them are termed *driving* members when they transmit motion, and *driven* members when they are elements to which motion is transmitted.

The ratio of the angular velocities of two shafts (pulleys or wheels) is called the velocity, or speed, ratio. The term *transmission ratio* is also used, and, for gears, *gear ratio.*

Since the angular velocity ω of a body in rad/s is proportional [formula (95)] to its angular velocity n in rpm, then if we denote the velocity ratio by the letter i with appropriate double index, we have

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{n_1}{n_2} \quad \text{or} \quad i_{21}=\frac{\omega_2}{\omega_1}=\frac{n_2}{n_1}. \qquad (106)$$

We will here confine the term *transmission ratio* to the *ratio of the angular velocity of the driving shaft to the angular velocity of the driven shaft* (in that order).

Let us examine how the velocity ratios of certain elementary-type drives are determined.

1. Belt drives

A belt drive between two parallel shafts may be an *open-belt* drive (Fig. 172*a*) or a *crossed-belt* drive (Fig. 172*b*).

Since the belt is drawn over the pulleys under tension, friction between the pulleys and the belt enables the driving pulley to set in motion the driven pulley by means of the belt.

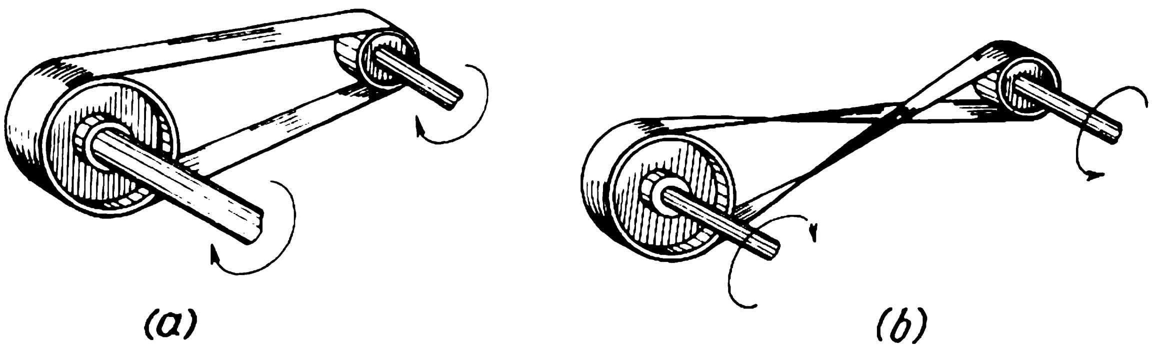

(*a*) (*b*)

FIG. 172

In the open-belt drive, the direction of rotation of the pulleys over which an endless belt is passed is the same.

In the crossed-belt drive, the direction of rotation of the driven pulley is opposite that of the rotation of the driving pulley. If we disregard slipping of the belt, the peripheral velocities of the pulleys are $v_1 = \omega_1 R_1$ and $v_2 = \omega_2 R_2$ and are equal, whence $\omega_1 R_1 = \omega_2 R_2$ and

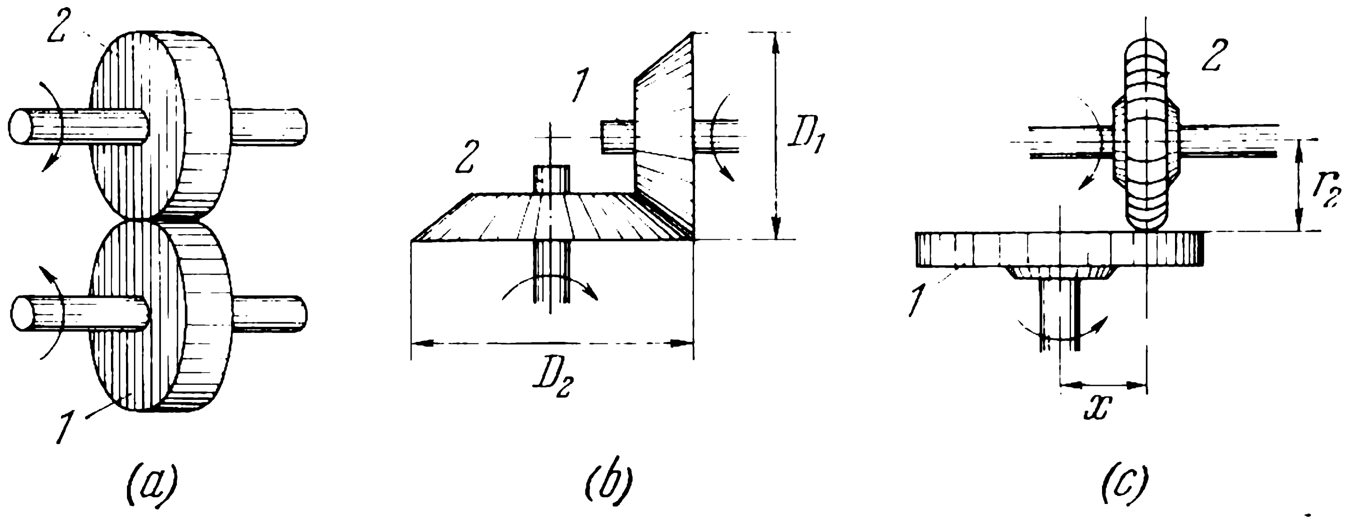
$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{R_2}{R_1}=\frac{D_2}{D_1}. \tag{107}$$

The velocity ratio of a belt drive varies inversely as the pulley radii (or diameters).

In actual practice, due to the unavoidable elastic slipping* of the belt when the belt drive is under load, the peripheral velocity v_2 of the driven pulley is from 1 to 3 per cent less than the peripheral velocity v_1 of the driving pulley; and to maintain the required angular velocity of the driven pulley, the diameter of the latter is reduced somewhat.

2. *Friction drives*

Friction drives are mechanisms in which the transmission of rotary motion is accomplished by wheels in contact. When the friction between the wheels is sufficient, the driving wheel turns and sets in motion the driven wheel.



(a) (b) (c)

FIG. 173

Figure 173*a* depicts a friction drive between two parallel shafts accomplished by cylindrical wheels, and Fig. 173*b* shows a drive between intersecting shafts accomplished by bevel friction wheels.

---

* This is one of the essential drawbacks of belt drives.

The velocity ratio of a friction drive is found from the condition that the velocities of the points in contact on the surfaces of the wheels must be equal (if, of course, slipping is disregarded).

The result is that the *velocity ratio of a cylindrical-wheel friction drive varies inversely as the radii (or diameters) of the wheels*:

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{R_2}{R_1}=\frac{D_2}{D_1}. \qquad (108)$$

The directions of rotation of the driving and driven wheels (Fig. 173*a*) are obviously opposite.

By agreeing, in the case of a bevel drive, to take the points in contact on the circles of the large bases of the cones, we find that the *velocity ratio of a bevel-wheel friction drive is also equal to the inverse ratio of the radii (diameters) of the circles of the large bases of the cones.*

For the plate-and-wheel variable-speed drive (Fig. 173*c*), which permits obtaining a variable velocity ratio between the driving shaft and driven shaft by means of a translational motion of wheel *2* along a diameter of wheel (plate) *1*, we have

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{r_2}{x},$$

where x is the variable distance of the middle plane of wheel *2* from the axis of wheel *1*, r_2 is the radius of wheel *2*.

3. *Gear drives*

The merits of friction drives are simplicity in design and smoothness in operation. A substantial drawback is the lack of constancy in the velocity ratio due to slipping of the friction wheels relative to each other. Slipping increases with increasing load transmitted by the wheels. The greater the force transmitted by the wheels, the stronger must be the contact between the wheels in order to ensure sufficient adhesion. And this in turn involves an increase in friction in the bearings and a considerable bending of the shafts.

To eliminate these drawbacks, the cylindrical surfaces of the wheels are supplied with teeth and spaces between them (Fig. 174). If one or several teeth of one wheel enter into the spaces of the other, then the teeth of the driving wheel, when rotating, press on the teeth of the other (driven) wheel and drive it. This ensures a constancy of the gear ratio of such a drive.

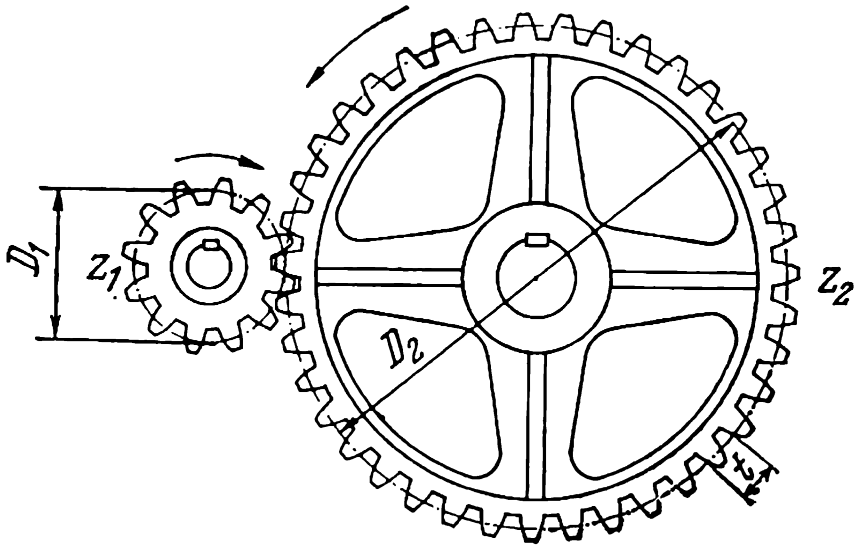


FIG. 174

Wheels with toothlike projections are called gear wheels, and the transmission of rotation by means of gear wheels is called a *gear drive*. Gear drives are one of the most common types of transmission mechanism.

The gear wheels depicted in Fig. 174 may be mentally replaced by two friction wheels rolling along each other (without slipping) and rotating on the same axes with the same angular velocities as the given gear wheels. The circumferences of such imaginary friction wheels are termed *pitch circles of the gear wheels.*

Consequently, *the gear ratio of two engaged gear wheels is inversely proportional to the radii of their pitch circles*:

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{R_2}{R_1}. \tag{109}$$

The gear ratio of a gear drive may be given a more convenient form for practical purposes.

The distance between two similar points on adjacent teeth (Fig. 174) *as measured on an arc of the pitch circle is termed circular pitch.*

The ratio of the circular pitch to the number π *is called the module*

$$m=\frac{t}{\pi}, \tag{110}$$

where t is the circular pitch and m is the module.

The pitch and the module are always measured in millimetres. The modules of gear wheels are standardized.

For the proper meshing of a gear set consisting of two wheels, it is necessary that their circular pitches (and, hence, modules) be the same. From this we obtain the following expressions for the lengths of the pitch circles of two engaged gear wheels:

$$\pi D_1 = tz_1 \quad \text{and} \quad \pi D_2 = tz_2,$$

where z_1 and z_2 are the numbers of the teeth of the corresponding wheels.

Hence, the diameters of the pitch circles of these wheels are

$$D_1=\frac{t}{\pi}z_1=mz_1 \quad \text{and} \quad D_2=\frac{t}{\pi}z_2=mz_2. \tag{111}$$

The diameter of the pitch circle of a gear wheel is equal to the product of its module into the number of teeth on the wheel.

Substituting the values of the diameters of the pitch circles of the wheels into formula (109), we get

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{z_2}{z_1}. \tag{112}$$

The gear ratio of two engaged gear wheels is equal to the inverse ratio of the number of their teeth.

Figure 175*a* shows a spur gear (*external gearing*) and Fig. 175*b* shows an internal gear (*internal gearing*). They serve to transmit rotation between parallel shafts.

It is sometimes convenient to regard the angular velocities of the gear wheels and the gear ratio between them as algebraic quantities. Taking the angular velocity of the driving wheel to be positive, we will assume the angular velocity of the driven wheel to be positive as well if it rotates in the same direction as the driving wheel, and negative if it rotates in the reverse direction.

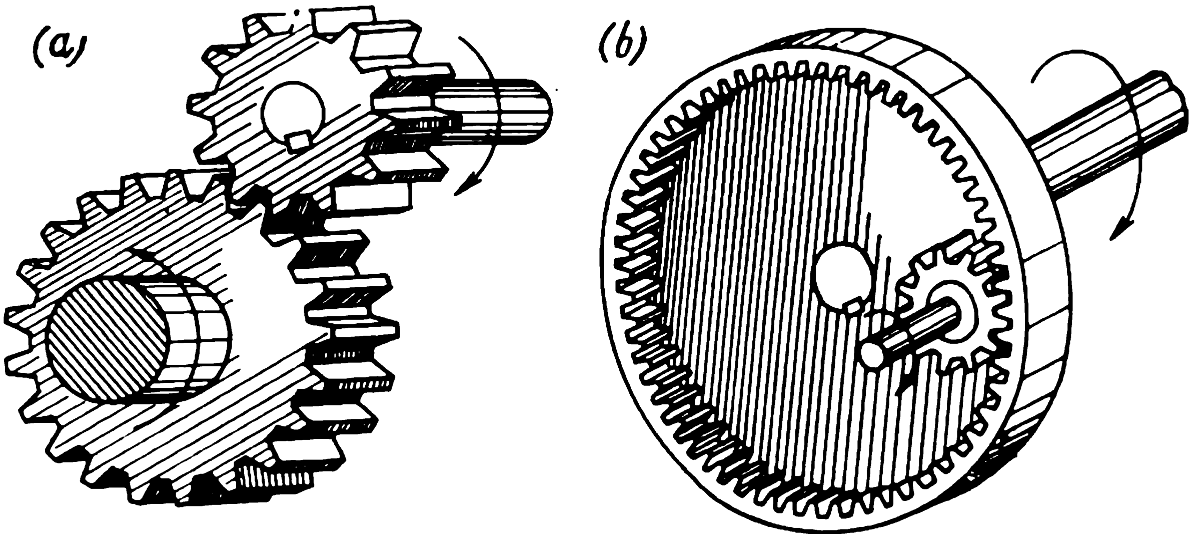


FIG. 175

As is evident from Fig. 175*a*, in *external gearing* the two engaged wheels rotate in opposite directions and so the *gear ratio* between them is

$$i_{12}=\frac{\omega_1}{\omega_2}=-\frac{R_2}{R_1}=-\frac{z_2}{z_1}.$$

Now in *internal gearing* (Fig. 175*b*), both wheels rotate in the same direction and the *gear ratio* between them is

$$i_{12}=\frac{\omega_1}{\omega_2}=\frac{R_2}{R_1}=\frac{z_2}{z_1}.$$

To transmit rotation between intersecting shafts, use is made of bevel gears (Fig. 176).

The gear ratio of two engaged bevel gears is determined, in absolute value, in the same way as for spur gears, by formula (112).

Since bevel gearing is not a plane mechanism but a spatial mechanism, the angular velocities of the bevel gears can no longer be regarded as algebraic quantities.

Sometimes a so-called *rack and pinion* combination is used to convert rotary motion into linear motion. This type of gear drive is frequently used in metal-cutting machine tools, for example, to move the table of a planing machine. In the lower part of the table is a toothed rack A (Fig. 177) meshed with a gear wheel B. The wheel is driven by the machine and sets the rack in linear motion and so also the table. Depending on the direction of rotation of the drive, the table of the machine performs a forward (or working) stroke and a reverse (or idle) stroke.

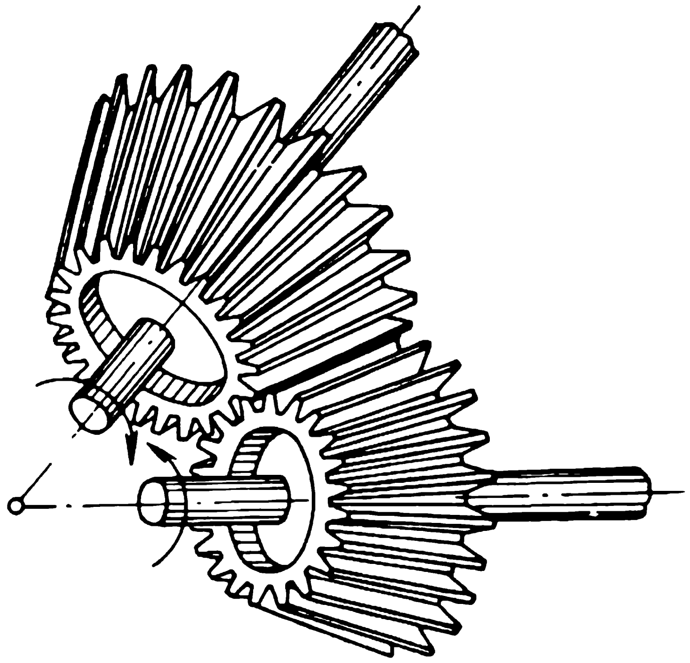


FIG. 176

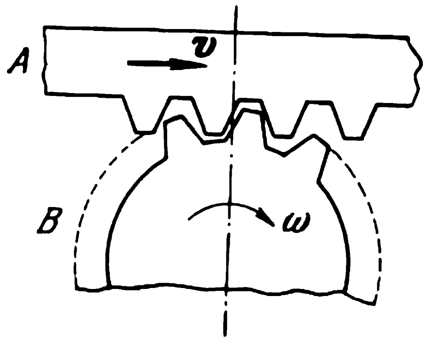


FIG. 177

The rate of linear motion of the rack is clearly equal to the velocity of the point lying on the pitch circle of the gear wheel: $v = \omega R_{\text{pitch}}$, where ω is the angular velocity of the wheel and R_{pitch} is the radius of its pitch circle, which radius may be found from formula (111).

4. *Multiple gearing*

Multiple gearing is the name given to a mechanism consisting of a number of interconnected simple gears.

Figure 178 depicts a scheme of multiple gearing from shaft I to shaft V which consists of a belt drive, two pairs of spur gears and one pair of bevel gears.

The pulleys and gears are rigidly mounted on the appropriate shafts (this is indicated by small crosses in the diagram). The radii of the pulleys are R_1 and R_2. The numbers of teeth in the gear wheels are denoted by z with appropriate subscripts.

The transmission ratio of the given multiple-gear drive is

$$i_{15}=\frac{\omega_1}{\omega_5}.$$

The transmission ratios of the simple drives that make up the given multiple-gear drive are

$$i_{12}=\frac{\omega_1}{\omega_2}, \quad i_{23}=\frac{\omega_2}{\omega_3}, \quad i_{34}=\frac{\omega_3}{\omega_4}, \quad i_{45}=\frac{\omega_4}{\omega_5}.$$

Multiplying these transmission ratios termwise, we get

$$i_{12}i_{23}i_{34}i_{45}=\frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3}\frac{\omega_3}{\omega_4}\frac{\omega_4}{\omega_5}=\frac{\omega_1}{\omega_5}=i_{15}.$$

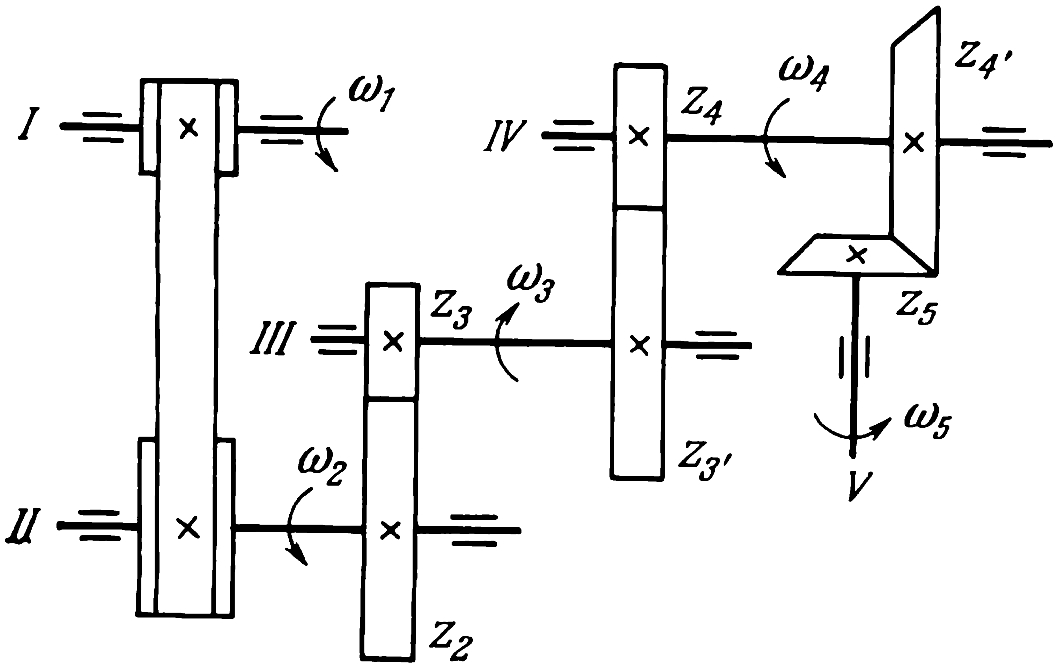


FIG. 178

We thus find that the *transmission ratio of this multiple-gear drive is equal to the product of the transmission ratios of all the constituent simple drives*:

$$i_{1n}=i_{12}i_{23}\ \dots\ i_{n-1,\,n}. \tag{113}$$

The transmission ratio of a multiple-gear drive may be computed if we know how the transmission ratios of the component simple drives are determined.

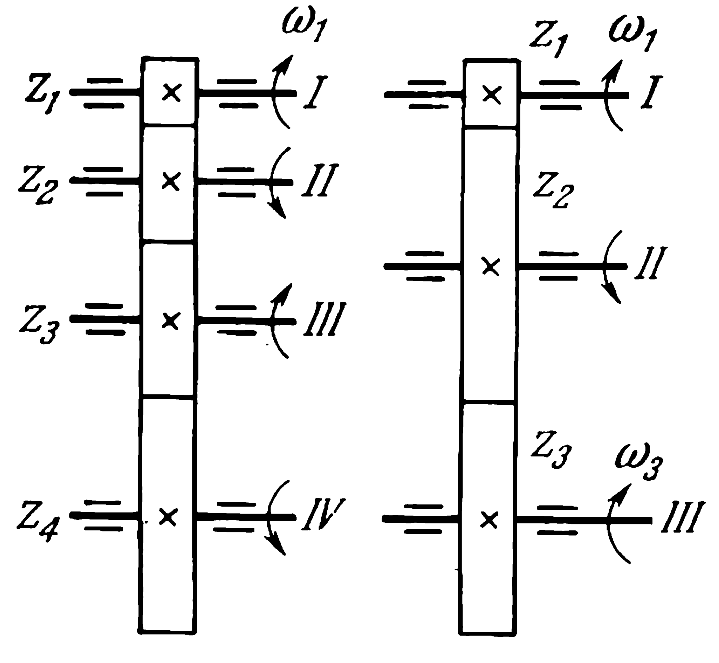
For the two-stage spur-gear drive from shaft *II* to shaft *IV* shown in Fig. 178, the transmission ratio is

$$i_{24} = i_{23} i_{34} = -\frac{z_3}{z_2}\left(-\frac{z_4}{z_{3'}}\right) = \frac{z_3 z_4}{z_2 z_{3'}}.$$

For the entire multiple-gear drive from shaft *I* to shaft *V*. the absolute value of the transmission ratio is

$$|i_{15}| = \left|\frac{\omega_1}{\omega_2}\right| = |i_{12}| \times |i_{23}| \times |i_{34}| \times |i_{45}| = \frac{R_2}{R_1}\frac{z_3}{z_2}\frac{z_4}{z_{3'}}\frac{z_5}{z_{4'}}.$$

A special case of multiple gearing is the so-called *gear train* shown in Fig. 179. In this arrangement, there is only one gear wheel on each shaft.



(a) (b)

FIG. 179

The transmission ratio for the set of spur gears in series shown in Fig. 179*a* is

$$i_{14} = \frac{\omega_1}{\omega_4} = i_{12} i_{23} i_{34} = -\frac{z_2}{z_1}\left(-\frac{z_3}{z_2}\right)\left(-\frac{z_4}{z_3}\right) = -\frac{z_4}{z_1}.$$

For the gearing shown in Fig. 179*b*, the transmission ratio is equal to

$$i_{13} = \frac{\omega_1}{\omega_3} = i_{12} i_{23} = -\frac{z_2}{z_1}\left(-\frac{z_3}{z_2}\right) = \frac{z_3}{z_1}.$$

As was mentioned earlier, the sense of rotation of the gear wheels in the case of internal gearing remains unchanged. In external gearing the gear wheels rotate in opposite directions. Thus, the sign of the transmission ratio of a compound gear train of spur gear wheels depends on the number of external engagements. For an even number k of external engagements (as for example for transmission from shaft *II* to shaft *IV* in Fig. 178 or for the transmission shown in Fig. 179*b*) the transmission ratio is positive, that is, the driven shaft rotates in the same direction as the driver. With an odd number k of external engagements (see Fig. 179*a*), the transmission ratio is negative, that is, the driven shaft rotates in the opposite direction to that of the driving shaft.

Consequently,

$$i_{1n} = (-1)^k |i_{1n}| = (-1)^k \frac{z_2}{z_1} \frac{z_3}{z_{2'}} \frac{z_4}{z_{3'}} \cdots \frac{z_n}{z_{(n-1)'}}, \qquad (114)$$

where k is the number of external gear engagements.

From the results of calculating the transmission ratios of gear trains it follows that the *presence of intermediate wheels in the gear train does not affect the absolute value of the transmission ratio.* That is why the *intermediate wheels of gear trains* are often called *idlers.*

In the case of any number of idlers and arbitrary dimensions of the idlers, *the transmission ratio of a gear train is equal, in absolute value, to the ratio of the number of teeth of the driven wheel to the number of teeth on the driving wheel.*

As we can see from the same results, the presence of idlers does not affect the absolute value of the transmission ratio but does affect the sign of the transmission ratio and, hence, the sense of rotation of the driven shaft.

The properties of idler wheels suggest the uses they are put to. They are used in transmission mechanisms in two cases:

1. When there is a substantial distance between the axes of the driving shaft and driven shaft. A direct connection between the driver and the driven shaft by a single pair of gears would in that case require gear wheels of too large a diameter.

2. When it is required to change the direction of rotation of the driven shaft without changing that of the driver. If only one intermediate (idler) wheel is introduced, the driven shaft rotates in the same direction as the driver. When there are two idlers, the driven shaft rotates in the opposite sense to that of the driver.

Problem 79. Referring to Fig. 180, pulley *II* is set into motion by an endless belt from pulley *I* of an engine. The diameters of the pulleys are: $D_1 = 25$ cm, $D_2 = 50$ cm; the angular acceleration of pulley *I* during the start-up period of the engine is constant and equal to $\varepsilon_1 = 2\pi$ rad/s^2. Determine the angular velocity (in rpm) of pulley *II* after a lapse of 15 seconds from start-up.

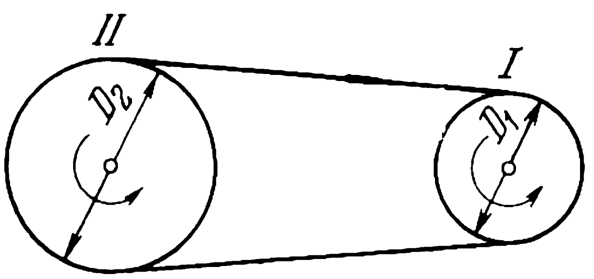


FIG. 180

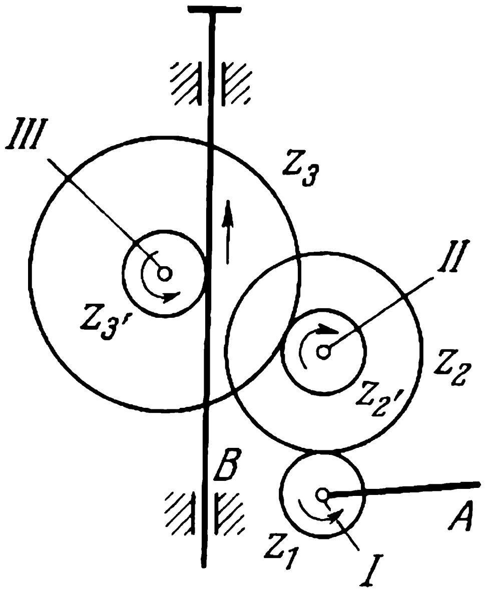


FIG. 181

Solution. Since the angular acceleration is constant, the angular velocity ω_1 of pulley *I* after 15 seconds from start-up of the engine will be $\omega_1 = \omega_0 + \varepsilon_1 t = 2\pi \times 15 = 30\pi$ rad/s.

The transmission ratio between the pulleys is

$$i_{21} = \frac{\omega_2}{\omega_1} = \frac{D_1}{D_2},$$

whence the angular velocity of pulley *II*, in 15 seconds, is

$$\omega_2 = \frac{\omega_1 D_1}{D_2} = \frac{30\pi \times 25}{50} = 15\pi \text{ rad/s}$$

or

$$n_2 = \frac{30\omega_2}{\pi} = \frac{30 \times 15\pi}{\pi} = 450 \text{ rpm.}$$

This problem could be tackled from a different angle. The ratio of the angular accelerations of the two shafts is equal to the velocity ratio between them: $\varepsilon_2/\varepsilon_1 = i_{21}$. Using this fact, we could have, without determining the angular velocity of pulley *I*, found the angular acceleration of pulley *II* from the velocity ratio and then the angular velocity of pulley *II* from the formula $\omega_2 = \omega_0 + \varepsilon_2 t$.

Problem 80. How many revolutions per minute must the handle *A* of a jack do (a kinematic diagram is shown in Fig. 181) so that the rack *B* rises at a speed of $v = 10$ mm/s? The wheels have the following numbers of teeth: $z_1 = 6$, $z_2 = 24$, $z_{2'} = 8$, $z_3 = 32$, $z_{3'} = 20$. The module of gear wheel $z_{3'}$ and the rack is $m = 4$ mm.

Solution. Using formula (111), we find the radius of the pitch circle of the gear wheel:

$$R_{\text{pitch}} = \frac{mz_{3'}}{2} = \frac{4 \times 20}{2} = 40 \text{ mm}.$$

Since the rate of climb of the rack is equal to the velocity of a point lying on the pitch circle of the gear meshed with the rack, the angular velocity of wheel $z_{3'}$ and shaft *III* (on which the wheel is mounted) is

$$\omega_3 = \frac{v}{R_{\text{pitch}}} = \frac{10}{40} = 0.25 \text{ rad/s}.$$

The absolute value of the transmission ratio of the multiple-gear drive of the jack is

$$i_{13} = \frac{\omega_1}{\omega_3} = i_{12} i_{23} = \frac{z_2}{z_1} \frac{z_3}{z_{2'}} = \frac{24}{6} \frac{32}{8} = 16.$$

The angular velocity of the driving shaft *I* and the handle mounted on it is

$$\omega_1 = \omega_3 i_{13} = 0.25 \times 16 = 4 \text{ rad/s}$$

or

$$n_1 = \frac{30\omega_1}{\pi} = \frac{30 \times 4}{3.14} \approx 38.2 \text{ rpm}.$$

CHAPTER XV

COMPOUND MOTION OF A POINT

Sec. 69. Absolute, relative and migratory motion of a point

In the introduction to kinematics we stated that any motion of a body or a point is relative motion, that is, it may be observed and studied only with respect to other physical bodies and associated frames of reference.

In the preceding chapters we considered motion with respect to a "fixed" frame of reference, which in engineering practice is taken to be a system of coordinates rigidly attached to the earth.

The motion of a point with respect to a reference frame assumed to be fixed is called absolute motion.

It is sometimes more convenient to regard the motion of a point relative to a fixed frame of reference as a compound motion consisting of two simultaneous motions: the motion of the point with respect to a certain moving coordinate system and the motion of the point together with that system relative to a fixed reference frame.

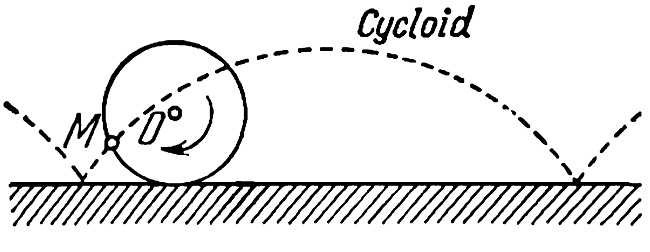


FIG. 182

For example, the motion of some point M on the wheel of a locomotive (Fig. 182) relative to the earth follows a curve called a cycloid and may be regarded as consisting of two simple motions: the motion of the point around the circumference of a circle relative to the body of the locomotive and the motion of that point together with the translational motion of the body of the locomotive.

The motion of a point relative to a moving reference frame is called relative motion.

The motion of a moving reference frame and of all points rigidly attached to it with respect to a fixed frame of reference is termed migratory motion.

In order to determine the migratory motion of some point at a given time, one has to mentally discard the relative motion of the point and, at that time, determine its motion with respect to a fixed system and as a point rigidly attached to the moving system.

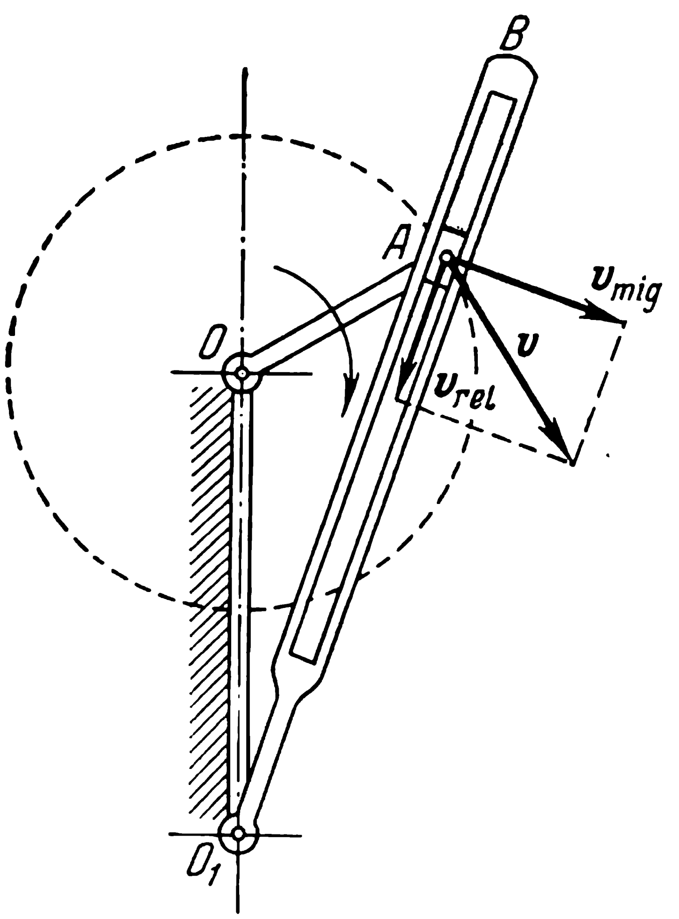


FIG. 183

This device is sometimes conveniently used to determine the relative motion of a point as well. To find that motion, mentally discard the migratory motion of the point.

In the illustration given above, the circular motion of point M with respect to the moving body of the locomotive is clearly relative motion. Now if we, mentally, attach this point rigidly to the body of the locomotive, then its motion together with the locomotive is migratory motion, whereas the motion of M (along the cycloid) with respect to the earth is absolute motion.

Here are some more illustrative examples. The motion of a person on the deck of a ship moving on a river is relative motion. The motion of the point on the deck of the ship at which the person stands at a given instant, relative to the shoreline, is migratory motion, and the motion of the person relative to the shoreline is absolute motion.

In the mechanism of a planing machine (Fig. 183), the absolute motion of point A (the centre of a hinge joining a slide block and the crank OA) is its motion relative to the fixed bed of the machine. Clearly, this is circular motion of point A about the fixed centre O. This motion may be resolved into components: the motion of point A along the slideway (a moving guide) O_1B is relative motion, and the rotational (oscillatory) motion about the centre O_1 of that point on the slideway O_1B that coincides at the given time with point A is migratory motion.

We stress once again that relative motion is motion with respect to a moving frame of reference, and absolute motion is the motion of a point relative to a fixed frame of reference. The terms "moving" and "fixed" as here applied to reference frames are of course used conditionally. For example, when considering the motion of a point on the wheel of a cart rolling along the deck of a ship, we can regard as fixed some coordinate system rigidly attached to the deck of the ship (which is in motion relative to the earth).

Let us agree now on the following notation: the absolute velocity of a point will be denoted by $\mathbf{v}$, the relative and migratory velocities by the same symbol with appropriate subscripts; $\mathbf{v}_{rev}$ for relative motion and $\mathbf{v}_{mig}$ for migratory motion.

The absolute velocity $\mathbf{v}$ *of a point is its velocity with respect to a fixed coordinate system.*

The relative velocity $\mathbf{v}_{rev}$ *of a point is its velocity with respect to a moving coordinate system.*

Somewhat more involved and requiring an explanation is the concept of migratory motion of a point in cases where the motion of the moving frame of reference is not translational motion.

For instance, in the example given above (Fig. 183), the migratory motion of point A (the centre of the sliding hinge) is the motion, together with slideway O_1B, of that point with which point A coincides at the given time. Due to the relative motion of the slide block along the slideway, the position of point A relative to the slideway will be changing all the time. But when the slideway turns about the fixed point O_1, different points on the slideway have different linear displacements and different linear velocities, and so the migratory velocity of point A will depend on the position that the point occupies at a given instant relative to the slideway.

Consequently, the *migratory velocity* $\mathbf{v}_{mig}$ *of some point* M *is the absolute velocity of that point (rigidly attached to a moving* *frame of reference) with which the given point M coincides at the given time.*

Since it is only in the case of translational motion of the moving reference frame that the velocities of all points associated with the system are the same, it follows that only in that case will the migratory velocity of the moving point be independent of its position relative to the moving frame, the migratory velocity being regarded as the velocity of the moving system with respect to the fixed system.

Sec. 70. Theorem of composition of velocities

Theorem. *The absolute velocity of a point is equal to the geometric sum of its migratory and relative velocities*:

$$\mathbf{v} = \mathbf{v}_{\text{mig}} + \mathbf{v}_{\text{rel}}. \qquad (115)$$

Proof. Suppose a point M is in motion relative to some moving reference frame S and is being displaced, together with that system, relative to a fixed reference frame $Oxyz$ (Fig. 184).

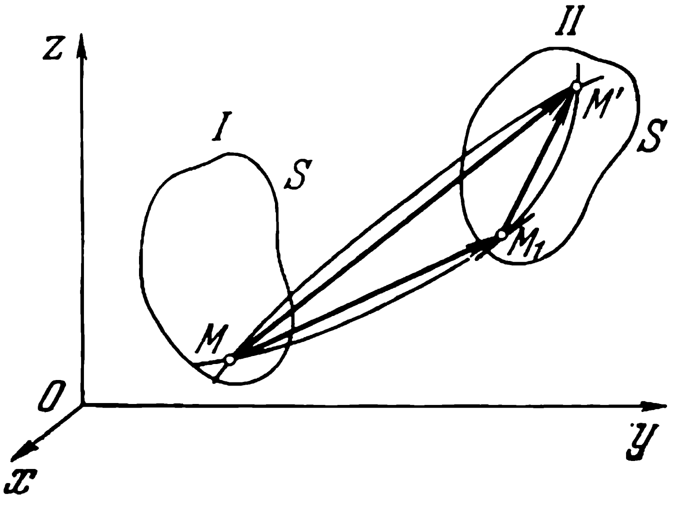


FIG. 184

Suppose the moving system S moves from position I to position II during the time interval Δt.

If the point M did not have relative motion, then it would move with respect to the fixed system $Oxyz$ along the arc $\overset{\frown}{MM_1}$ of a certain trajectory from M to M_1, all the while occupying a fixed position relative to the moving system S.

The vector $\overrightarrow{MM_1}$ may be called* the vector of migratory displacement of point M for the given time interval Δt. Due to the relative motion of M, this point will move

---

* Recall (see Sec. 55) that the displacement vector of a point during a given time interval Δt is a vector that joins the positions of the point at the beginning and at the end of that time interval.

(during the given time interval) relative to the moving system S along the arc $\overset{\frown}{M_1M'}$ of the trajectory of its relative motion and will take up a certain position M'.

Quite obviously, the vector $\overrightarrow{M_1M'}$ may be termed the vector of relative displacement of point M over the given time interval.

Actually, both motions (migratory and relative) take place at the same time and the point M, during the time interval Δt, moves from M to M' and is displaced relative to the fixed system $Oxyz$ along a certain arc $\overset{\frown}{MM'}$ of the *trajectory of its absolute motion. Hence, the vector* $\overrightarrow{MM'}$ is the vector of the absolute displacement of the point.

From Fig. 184 it is evident that

$$\overrightarrow{MM'} = \overrightarrow{MM_1} + \overrightarrow{M_1M'}.$$

Dividing both sides of the equality by the same quantity Δt (which leaves the vector equation unchanged), we get

$$\frac{\overrightarrow{MM'}}{\Delta t} = \frac{\overrightarrow{MM_1}}{\Delta t} + \frac{\overrightarrow{M_1M'}}{\Delta t}.$$

But as we know, the ratio of the displacement vector of a point (in the motion of the point) to the time interval during which the displacement is carried out yields the average-velocity vector of the point over the given time interval. Hence

$$\frac{\overrightarrow{MM'}}{\Delta t} = \mathbf{v}_{\text{av}}, \qquad \frac{\overrightarrow{MM_1}}{\Delta t} = \mathbf{v}_{\text{av, mig}}, \qquad \frac{\overrightarrow{M_1M'}}{\Delta t} = \mathbf{v}_{\text{av, rel}}.$$

Thus we obtain

$$\mathbf{v}_{\text{av}} = \mathbf{v}_{\text{av, mig}} + \mathbf{v}_{\text{av, rel}}.$$

If we diminish the time interval Δt without bound, the points M_1 and M' will approach the point M without bound, and the chords $\overrightarrow{MM'}$, $\overrightarrow{MM_1}$ and $\overrightarrow{M_1M'}$ will approach without bound the tangents to the curves MM', MM_1 and M_1M' at the point M. In the limit, as $\Delta t \to 0$, the average velocities of the point over the time interval Δt will become the appropriate true velocities of the point at the given time t, the velocities being along the tangents to the appropriate curves MM', MM_1, M_1M' (Fig. 184). Thus, passing to the limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \mathbf{v}_{\mathrm{av}} = \lim_{\Delta t \to 0} \mathbf{v}_{\mathrm{av,\,mig}} + \lim_{\Delta t \to 0} \mathbf{v}_{\mathrm{av,\,rel}}$$

or

$$\mathbf{v} = \mathbf{v}_{\mathrm{mig}} + \mathbf{v}_{\mathrm{rel}}.$$

The proof of the theorem is complete.

Since in the geometric composition of two velocities of a point the resulting velocity is depicted as a diagonal of a parallelogram constructed on the component velocities as sides (Fig. 185) or, what is the same thing, as the closing side of a vector triangle, the given theorem is often termed the *parallelogram rule*.

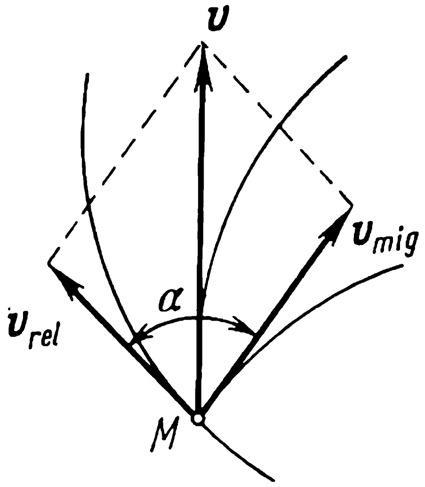


FIG. 185

If the magnitudes of the migratory $\mathbf{v}_{\mathrm{mig}}$ and relative $\mathbf{v}_{\mathrm{rev}}$ velocities of a point and the angle α between their directions are known, then the magnitude of the absolute velocity is found by the cosine theorem (in exactly the same way as for the composition of two forces applied to a single point—see Sec. 8):

$$v = \sqrt{v_{\mathrm{mig}}^2 + v_{\mathrm{rel}}^2 + 2v_{\mathrm{mig}}v_{\mathrm{rel}} \cos \alpha}. \qquad (116)$$

If the directions of the migratory and relative motions of the point are perpendicular to each other, then $\alpha = 90°$, $\cos \alpha = 0$ and $v = \sqrt{v_{\mathrm{mig}}^2 + v_{\mathrm{rel}}^2}$. If the migratory and relative motions are along a single straight line in the same direction, then $\alpha = 0$, $\cos \alpha = 1$, and $v = v_{\mathrm{mig}} + v_{\mathrm{rel}}$.

If the migratory and relative motions are along one straight line in opposite directions, then $\alpha = 180°$, $\cos \alpha = -1$ and $v = |v_{\mathrm{mig}} - v_{\mathrm{rel}}|$, that is, in this case the absolute velocity of the point is equal in absolute value to the absolute value of the difference between the migratory and relative velocities of the point and is in the direction of the greater velocity.

Problem 81. A motor vessel is in rectilinear motion with a constant velocity v_1. A person on the deck throws a ball vertically upwards with an initial velocity v_0.

Find the absolute trajectory of the ball and its absolute velocity (air resistance is ignored).

Solution. With respect to a fixed reference frame, the ball participates in two rectilinear motions: relative (with respect to the deck of the vessel) vertical motion under the force of gravity, and migratory horizontal uniform motion together with the deck of the vessel.

For the origin of the fixed coordinate system we take the fixed point corresponding to the place on the deck from which the ball was thrown; from that point we draw the axes: the x-axis horizontally in the direction of motion of the vessel, and the y-axis vertically upwards. With this choice of reference frame, the equation of motion of the ball (rigidly attached to the vessel)—the equation of the uniform motion of the vessel—is

$$x = v_1 t.$$

The equation of the relative motion of the ball (the equation of its motion with respect to the moving reference frame; here, the ship's deck) is, by (89), $y = v_0 t - \frac{gt^2}{2}$. At every instant of time t, these equations give the coordinates of the ball relative to a fixed frame of reference. In order to determine the flight-path equation of the ball relative to this reference frame, it is necessary to eliminate the time t from the given equations. From the first equation we have $t = x/v_1$. Substituting this value of t into the second equation, we obtain

$$y = \frac{v_0}{v_1} x - \frac{g}{2v_1^2} x^2.$$

Hence the absolute trajectory of the ball is a parabola defined by the given equation; the migratory velocity of the ball is

$$v_{\text{mig}} = v_1;$$

by formula (88), the relative velocity of the ball is

$$v_{\text{rel}} = v_0 - gt;$$

thus the absolute velocity of the ball is

$$v = \sqrt{v_{\text{mig}}^2 + v_{\text{rel}}^2} = \sqrt{v_1^2 + (v_0 - gt)^2}.$$

Problem 82. Referring to Fig. 186, a cam in the form of a wedge ABC with angle α is in translational motion along a horizontal plane with velocity $\mathbf{u}$. Find the velocity of the rod (push rod) DE resting on the cam and freely sliding in a fixed sleeve.

Solution. When the wedge moves to the right, the rod DE moves translationally and vertically upwards. Since in translational motion all points have the same velocities, we will find the velocity of one point, D. If we take the wedge for the reference frame, then with respect to that frame the point D moves on the straight line BC. Since the wedge itself is also in motion relative to a fixed reference frame, the motion of D with respect to the wedge is relative motion, but the motion of D together with the wedge is migratory motion.

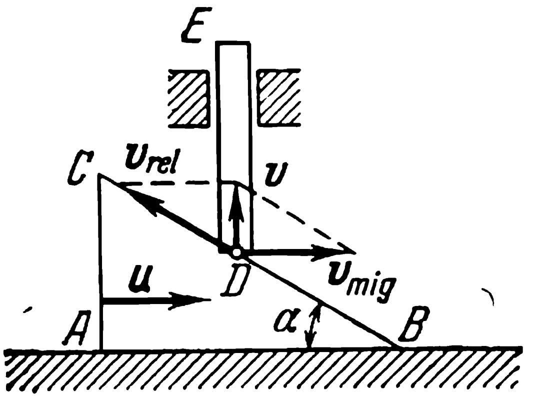


FIG. 186

The directions of the relative velocity $\mathbf{v}_{\mathrm{rel}}$ and the absolute velocity $\mathbf{v}$ of point D are known. The presence of a fixed sleeve permits the rod (and its point D) to have only one motion with respect to the fixed reference frame, that is, vertical motion.

Since the wedge is in translational motion, the migratory velocity of D is known both in magnitude and in direction, $v_{\mathrm{mig}} = u$. We now construct a velocity parallelogram of point D.

From the end of vector $\mathbf{v}_{\mathrm{mig}}$ (migratory velocity of D) we draw a straight line parallel to the relative velocity (the straight line BC) up to intersection with DE, along which the absolute velocity is directed.

From the right triangle of velocities we find

$$v = v_{\mathrm{mig}} \tan \alpha = u \tan \alpha.$$

Problem 83. Referring to Fig. 187, cranks O_1A and O_2B of the axles O_1 and O_2 of a locomotive are of length $r = O_1A = O_2B = 1$ m, are revolving with constant angular velocity $\omega = 10$ rad/s, and are connected by a connecting rod AB. The wheels have radii $R = 1.5$ m. The wheels roll on the rails without sliding. Find the absolute velocity of any point M of connecting rod AB for the four positions of the crank shown in Fig. 187: A_1, A_2, A_3 and A_4.

Solution. Since AB performs translational motion, the velocities of all its points are the same. The velocity of M is equal to that of A (points at which the end of the connecting rod is hinged to crank O_1A). The point A may be regarded as participating in two motions: a relative (rotational) motion about the axle O_1 of the locomotive and a migratory motion together with the wheel in its translational motion with a velocity equal to v_0 of the axle O_1 of the wheel. We shall find the velocity v_0 of the axle O_1 of the locomotive or, what is the same thing, the velocity of the locomotive. Let us consider the motion of point C of the wheel at which it touches the rail. This point too participates in two motions: migratory with velocity $v_{\mathrm{mig}} = v_0$ and relative (rotational) about the axle O_1. The relative velocity $\mathbf{v}_{\mathrm{rel}}$ of the point must obviously be directed (as shown in Fig. 187) along the tangent to the circumference of the wheel in the direction of revolution and must be equal in magnitude to $v_{\mathrm{rel}} = R\omega$.

But the wheel rolls without slipping and so the absolute velocity of the point of contact of the wheel and rail must be zero. Then, since the migratory and relative motions of C are along one straight line in opposite directions, we have

$$v = v_{\text{mig}} - v_{\text{rel}} = v_0 - R\omega = 0$$

or

$$v_0 = R\omega = 1.5 \times 10 = 15 \text{ m/s}.$$

The relative velocity of point A (the linear velocity of its rotation about the axle), $v_{\text{rel}} = r\omega = 1 \times 10 = 10$ m/s = constant. But the direction of this velocity varies depending on the position of point A. The migratory velocity of this point is $v_{\text{mig}} = v_0$. Now let us construct (Fig. 188) a velocity parallelogram for the various positions of the crank and from them let us find the magnitude and direction of the absolute velocity of point A:

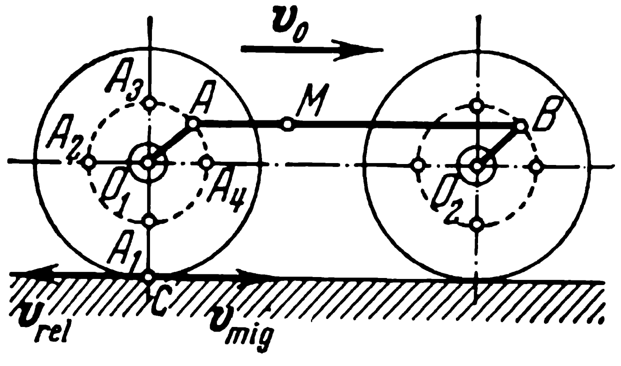


FIG. 187

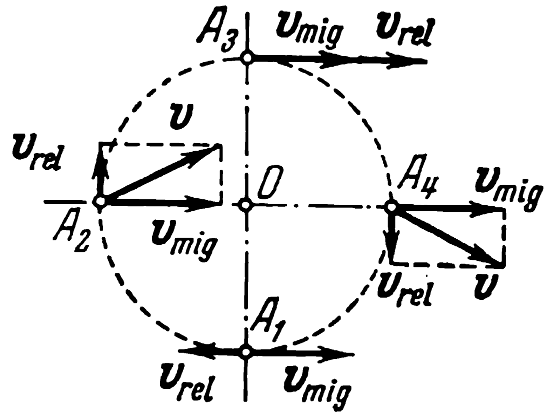


FIG. 188

(1) $v = v_{\text{mig}} - v_{\text{rel}} = 15 - 10 = 5$ m/s;

(2) $v = \sqrt{v_{\text{mig}}^2 + v_{\text{rel}}^2} = \sqrt{15^2 + 10^2} \approx 18$ m/s;

(3) $v = v_{\text{mig}} + v_{\text{rel}} = 15 + 10 = 25$ m/s;

(4) $v = \sqrt{v_{\text{mig}}^2 + v_{\text{rel}}^2} = \sqrt{15^2 + 10^2} \approx 18$ m/s.

Sec. 71. Resolving the velocity of a point into components

Very often the absolute velocity of a point is known and one has to find its components: that is, one has to decompose the absolute velocity into components. Just as the problem of the composition of velocities is similar to that of the composition of forces applied to a point, so the inverse problem of resolving absolute velocity of a point into migratory and relative velocities is completely analogous to the problem of decomposing a force into two component converging forces (Sec. 9). The solution of these problems is correct when the absolute velocity is a diagonal of a parallelogram constructed on the vectors of the migratory and relative velocities of the point. Since an infinity of parallelograms may be constructed on a given diagonal, it follows that, like the problem of decomposing a force, the problem of resolving the velocity of a point is indeterminate in the general case. To make the solution determinate requires specifying two supplementary conditions (either the directions of the component velocities or the magnitude and direction of one of them, and so forth).

Problem 84. How long will a passenger in a train moving at 72 km/h be able to see a train coming from the opposite direction if the second train has a velocity of 54 km/h and the length $l = 175$ m?

Solution. Placing a moving reference frame in the second train, we find the relative velocity of the first train; this is the velocity of the first train relative to the second train. We know the absolute velocity of the first train, that is, its velocity with respect to a fixed reference frame (the rails): $v = 72$ km/h and its migratory velocity, that is, the translational velocity of the moving system (of the second train) with respect to the fixed system: $v_{\mathrm{mig}} = 54$ km/h. Both motions are along a straight line in opposite directions. Therefore,

$$v = v_{\mathrm{rel}} - v_{\mathrm{mig}},$$

whence

$$v_{\mathrm{rel}} = v + v_{\mathrm{mig}} = 72 + 54 = 126 \text{ km/h} = \frac{126 \times 1000}{3600} = 35 \text{ m/s}.$$

The time during which a passenger in the first train will be able to watch the train coming in the opposite direction is

$$t = \frac{l}{v_{\mathrm{rel}}} = \frac{175}{35} = 5 \text{ s}.$$

Problem 85. Referring to Fig. 189, a rectilinear slider of a crank and slider mechanism of a power hammer performs reciprocating motion. The slider is set in motion by a slide block A joined to the end of crank OA of length $r = 30$ cm; the crank rotates uniformly with an angular velocity of $n = 150$ rpm. At $t = 0$ the slider is in its lowest position. Find the rate of the hammer (slider) at time t.

Solution. We express the angular velocity of the crank in radians per second: $\omega = \frac{\pi n}{30} = \frac{\pi \times 150}{30} = 5\pi$ rad/s. We will reckon the angle of rotation of the crank from its lowest vertical position. Then at $t = 0$, we have $\varphi_0 = 0$. Hence, at time t, the angle of rotation of the crank is $\varphi = \omega t = 5\pi t$. The absolute velocity of point A (the centre of the hinge connecting the crank and slide block) is equal in magnitude to $v = r\omega = 30 \times 5\pi = 150\pi$ cm/s and is directed perpendicularly to the radius of the crank in the direction of its rotation.

On the other hand, point A may be regarded as participating in two motions: a relative motion (horizontal) of the slide block along the moving guide (slideway) and a migratory motion together with the vertical translational motion of the slider.

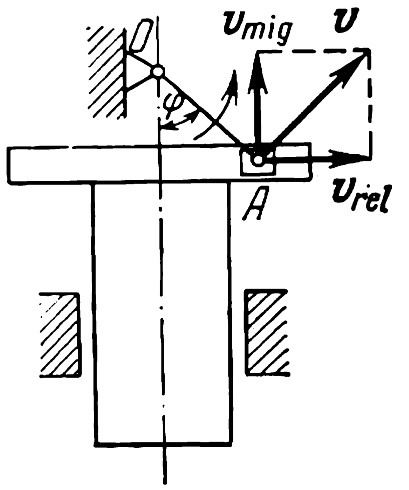


FIG. 189

We construct a parallelogram (see Fig. 189) whose diagonal is the vector **v** of absolute velocity of A (we know the magnitude and direction of this vector), and the sides are the vectors of relative, $\mathbf{v}_{rel}$, and migratory, $\mathbf{v}_{mig}$, velocities of that point (these are known in direction alone).

From the given parallelogram (rectangle) we find the velocity of the hammer (slider) which for A serves as the migratory velocity:

$$v_{mig} = v \sin\varphi = 150\pi \sin 5\pi t \text{ cm/s}.$$

CHAPTER XVI

COMPOUND MOTION OF A RIGID BODY

Sec. 72. The concept of compound motion of a body

The concept of the compound motion of a body is similar to that of the compound motion of a point. In certain cases, the motion of a body with respect to a fixed reference frame is conveniently regarded as compound motion consisting of two motions: relative motion, that is the motion of the body with respect to some moving frame of reference, and migratory motion, the motion of the body together with the moving reference frame with respect to a fixed system.

Every compound motion of a body may be reduced to some collection of translational and rotational motions, which are not only the simplest but also the basic forms of motion of a rigid body. The problem of finding the absolute motion of a body ordinarily reduces to that of the composition either of translational motions or rotational motions, or a rotational and a translational motion, depending on which the migratory and relative motions of the body are. In this chapter we consider a few special cases of particular practical significance of such a composition of motions of a body.

We stress that here we will be determining only the so-called *instantaneous absolute motion* of a body; that is, we will be seeking ways to find the absolute velocities of its points at a given instant of time.

Sec. 73. Plane-parallel motion of a body

Plane-parallel (or plane) motion of a rigid body is motion in which all points of the body move in planes parallel to a certain fixed plane.

A particular instance of such motion is the rotation of a rigid body about a fixed axis, which we have already studied. As we know, in rotational motion, all points of the body move in planes perpendicular to the axis of rotation and, hence, any one of the planes may be taken as the fixed plane parallel to which all points of the body move.

In a number of cases, plane motion is at the same time translational motion. However, one cannot, generally speaking, regard translational motion as a special case of plane motion. Not every translational motion is plane motion; likewise, not every plane motion of a body is translational motion.

Imagine a prism whose base can move at random over a fixed plane (Fig. 190). In this motion of the prism, all its points will obviously move in planes parallel to the fixed plane and therefore the prism is in plane motion.

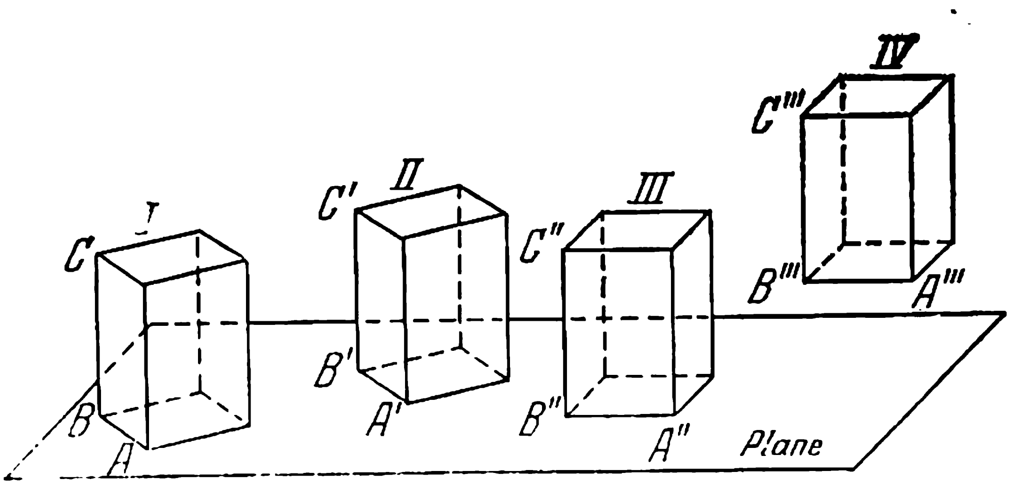


FIG. 190

Again referring to Fig. 190, if the prism moves from position *I* to *II*, then any straight line connected rigidly with it will all the time remain parallel to itself and the motion of the prism is plane and translational. Now, when the prism moves from position *II* to position *III*, as is clearly seen in the drawing, not every straight line connected with the prism (say, *AB*) remains parallel to itself, and the motion of the prism during this time interval is no longer translational, though it does remain plane motion. When the prism moves from position *III* to position *IV* (Fig. 190), the bases and lateral faces of the prism move into other planes, and the motion of the prism is no longer plane, although it could be translational if during the motion every straight line rigidly attached to the prism remained parallel to itself.

Plane motion is a very common motion in engineering practice. The vast majority of mechanisms are plane, that is to say, they constitute a combination of rigid bodies that perform plane motions.

Such for example is the motion of all members of the crank and slide mechanism shown in Fig. 191. The mechanism consists of a crank *OA*, a slide block *B* and a connecting rod *AB* hinged to it. All points of each of the members move in parallel to a certain fixed plane (the plane of the drawing of Fig. 191). The plane motion of the crank is at the same time also a rotational motion about a fixed axis *O*. The plane-parallel motion of the slide block is at the same time a translational motion along fixed guideways. Now, the plane motion of the connecting rod is neither rotational (because the connecting rod does not have any fixed points) nor translational (since the straight line *AB* does not remain parallel to itself during motion of the rod).

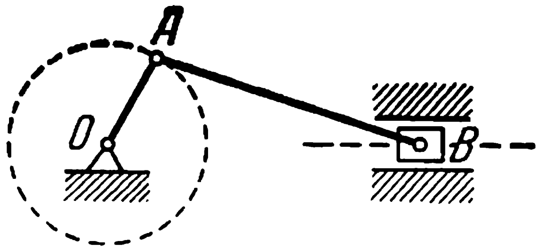


FIG. 191

Instances of plane-parallel motion are: a wheel rolling on a rectilinear rail,* the connecting-rod mechanism of locomotive wheels and the crank and slider that were both considered in problems 83 and 85.

Let us now investigate ways of simplifying the study of this very important form of motion of a rigid body. Suppose a body is in motion parallel to a certain fixed plane *Pl* (Fig. 192). If we intersect the given body with a plane *Pl′* parallel to the fixed plane *Pl*, then the section yields a plane figure *S*. This figure will move with the body and will remain in the same plane *Pl′*. Clearly, in this motion, all the points of the body lying on the perpendicular *Aa* erected to the plane figure *S* at some point *a* move in the same way, just as point *a* of that figure. All points lying on the perpendicular Bb move in the same manner as point b of figure S, and so on.

---

* In this case, all points of the wheel move in planes parallel to a fixed vertical plane passing through the midline of the rail. When there is a curve in the rails, the motion of the wheel is no longer plane motion.

From this it follows that *to determine the plane motion of a body it suffices to know the motion of an invariable plane figure obtained in the section of the body cut by a plane parallel to the given fixed plane.*

The study of plane motion can thus be replaced by the study of the motion of that plane figure in its plane.

Also note that the position, in the plane, of an invariable plane figure is fully determined by the position of any two of its points or, what is the same, by the position of some line segment rigidly attached to the moving figure. Suppose (Fig. 193) the rigidly attached line CD of this figure took up position $C'D'$ in the same plane. Since the distances DE and CE of any point E of the figure from the given points D and C are invariable, the new position of the point is readily found by the construction of a triangle $D'E'C'$ congruent to DEC.

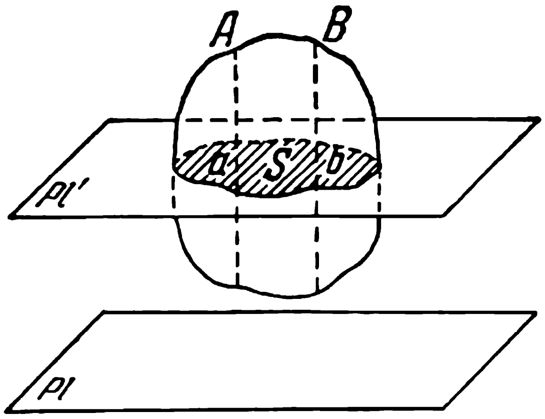


FIG. 192

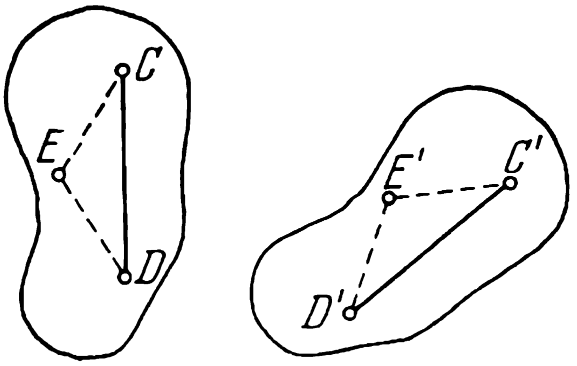


FIG. 193

Sec. 74. Resolving the motion of a plane figure into translational motion and rotational motion. The relationship between the velocities of distinct points of the figure

Referring to Fig. 194, suppose an arbitrary straight line rigidly attached to the plane figure moves with that figure over a certain time interval from position AB to position $A'B'$.

This displacement of the plane figure may be imagined as made up of a translational and a rotational displacement (Fig. 194*a*). Indeed, the displacement of straight line AB to position $A'B'$ could be obtained by a translational displacement to position $A'B''$ or $A''B'$ and a rotational displacement of that straight line about the axis passing respectively through point A' or point B' and perpendicular to the plane of the figure.

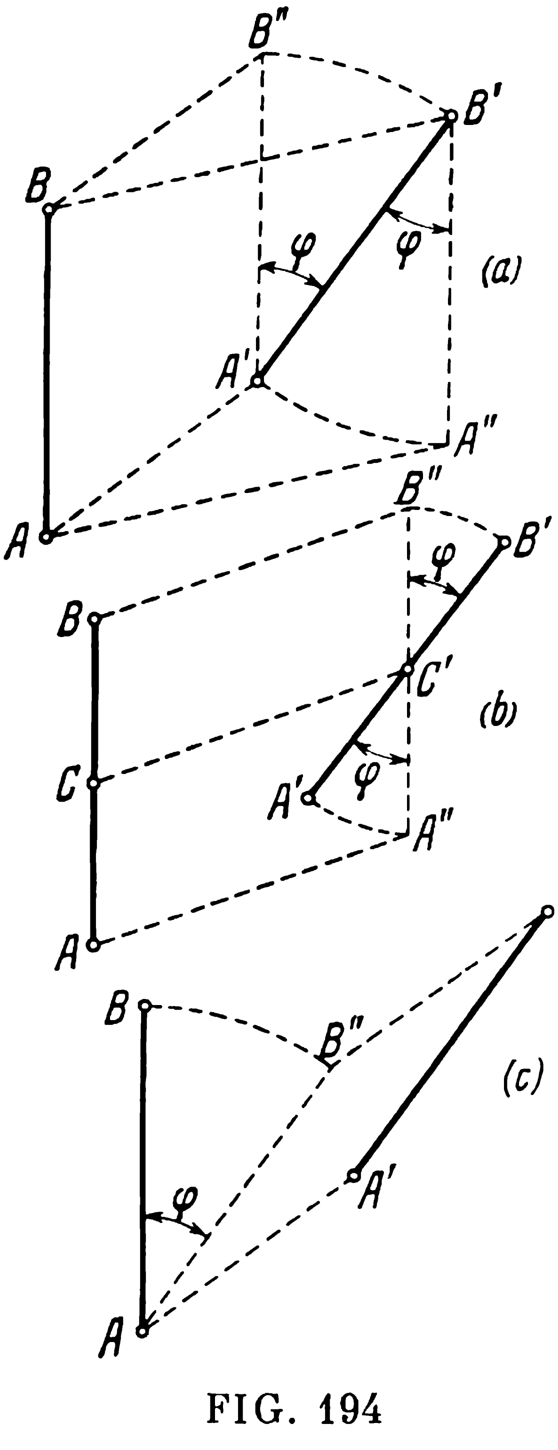


FIG. 194

The arbitrary point connected with the moving figure and taken as the centre of its rotation is termed a *pole*. It will readily be seen that by choosing different poles we merely change the translational displacement of the figure, while the angle of rotation and the sense of rotation of the figure do not depend on the choice of pole.

Thus, in the translational displacement of the line AB to position $A'B''$, all points of the figure execute displacements equal to the actual displacement of point A of that figure, and in the translational displacement of AB to position $A''B'$, all points of the figure execute displacements equal to the actual displacement of point B of the figure. But when AB goes to $A'B'$, the displacements of points A and B are, generally, not equal. Now, the angle of rotation $B''A'B'$ executed by $B''A'$ in moving to position $A'B'$ by rotation about A' is equal to the angle (they are alternate angles) of rotation $A''B'A'$ executed by $A''B'$ when it moves to the same position $A'B'$ via rotation about point B'.

In this process, the sense of rotation of the lines $B''A'$ and $A''B'$ is the same (clockwise in Fig. 194).

We obtain the same result if we take any other point for the pole, say point C (Fig. 194*b*), attached to the moving figure, or if we first rotate the line AB about one of its points through an appropriate angle φ and then give it a translational displacement (Fig. 194*c*).

To summarize: *any displacement of a plane figure in its plane may be achieved by a combination of a translational displacement of the figure equal to a displacement of some arbitrarily chosen point of it (the pole), and a rotational displacement of the figure about an axis that passes through the chosen pole and is perpendicular to the plane of the figure.*

This proposition holds true both for finite displacements and for infinitesimal displacements of a plane figure.

Note that in a finite displacement of a figure, that is, a displacement of the figure over a finite time interval, such a replacement of the actual compound motion of the figure by two successive (translational and rotational) displacements does not, generally speaking, reproduce that motion. This replacement merely produces the same final result, that is, the same new position of the figure. In the interval between the positions under consideration the figure could occupy any positions in the plane that do not at all coincide with the positions resulting from separate execution of translational and rotational displacements.

In order to reproduce the actual motion of the figure during a certain time interval, let us partition it into a large number of very small time intervals and note those positions that the figure occupies at the end of each of the time intervals. Choosing an arbitrary point of the figure for the pole, we then move the figure from each given position to a close-lying adjacent position by means of a translational displacement of the figure corresponding to the translational displacement of the chosen pole, and a rotational displacement of the figure about the axis passing through the pole perpendicular to the plane of the figure. By constantly increasing the number of time intervals and thus diminishing their duration, we will carry the figure through an ever larger number of positions which it actually occupies at the end of the given small time intervals. In the limit, when the size of the time intervals tends to zero, we will be carrying the figure through all positions that it actually occupies in its motion. We thus conclude that *any motion of a plane figure in its plane may be decomposed into two motions*: (1) *translational motion together with an arbitrarily chosen point of the figure (the pole)*, *and* (2) *rotational motion about that point.*

It was shown above that the angle φ of rotation of a figure and the direction of its rotation do not depend on the choice of pole.

From this it also follows that the angular velocity

$$\omega = \frac{d\varphi}{dt}$$

is independent of the choice of pole and, in the case of plane motion, is a kinematic characteristic common to all points of the body.

Suppose a certain plane figure S is in motion in its plane (Fig. 195). Take an arbitrary point A of the figure for the pole (Fig. 195a). Then, as has already been established, it may be considered that relative to some fixed system (attached to the plane in which the figure is moving), any other point B of the figure participates simultaneously in two motions: migratory motion (together with the figure in its translational motion with velocity $\mathbf{v}_A$ of the chosen pole) and relative motion (a rotational motion about the pole A with angular velocity ω that is independent of the choice of pole).

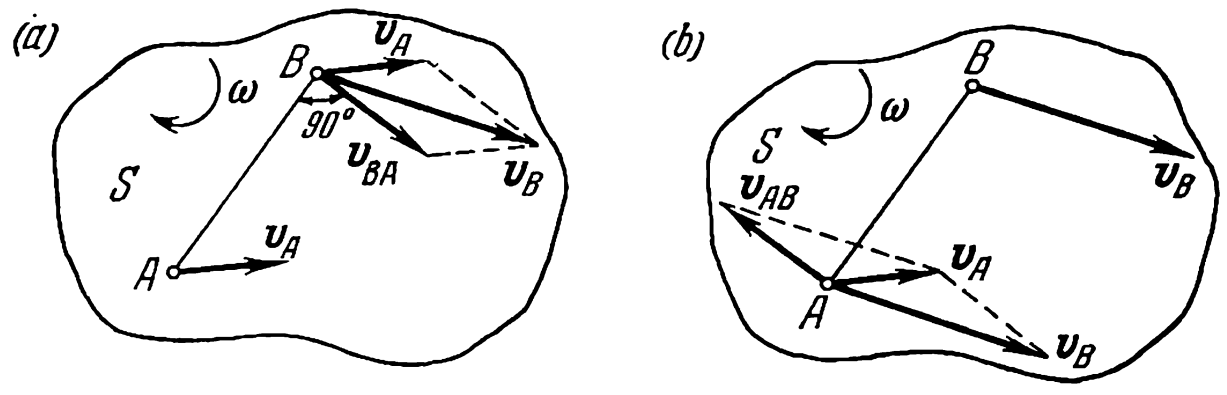


FIG. 195

From this, on the basis of the theorem on the composition of velocities, we have that the absolute velocity or (as we henceforth call it) merely the *velocity of any point of a plane figure at every given instant is equal to the geometric sum of two velocities: the velocity of another (arbitrarily chosen) point of the figure (the pole) and the rotational velocity of the former point with respect to the latter*:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}, \tag{117}$$

where $\mathbf{v}_B$ is the velocity of any point B of the plane figure, $\mathbf{v}_A$ is the velocity of another arbitrary point (pole) A of the figure, $\mathbf{v}_{BA}$ is the rotational velocity of the former point relative to the latter point (pole).

The rotational velocity of point B with respect to point A is easily found in magnitude and direction if we know (for the given time) the angular velocity of the figure and the positions of points B and A. The magnitude of this velocity is $v_{BA} = \omega \times BA$, where ω is the angular velocity of the figure and BA is the distance between points B and A. Like every rotational velocity, the velocity $\mathbf{v}_{BA}$ is directed perpendicularly to the appropriate radius of rotation (that is, to the line BA) in the direction of rotation of the figure (in Fig. 195, the arrow indicates clockwise rotation).

Having determined the rotational velocity $\mathbf{v}_{BA}$ of point B about the pole and knowing the velocity $\mathbf{v}_A$ of the pole itself, we can find the desired velocity of B as the diagonal of a parallelogram constructed on the vectors $\mathbf{v}_A$ and $\mathbf{v}_{BA}$ of the component velocities (Fig. 195*a*).

Note that the formula (117) establishes the relationship between the velocities of any two points of a plane figure, either of which may be chosen as the pole. Ordinarily, that point of the figure whose velocity is known at a given time is chosen as the pole. For instance, if in Fig. 195 we know the velocity of point B and it is required to find the velocity of A, then taking B as the pole, we would write

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}.$$

In this equation the rotational velocity of A about B will, as before, be equal to the magnitude of $v_{AB} = \omega \times \times AB$. But it will be in the opposite direction to $\mathbf{v}_{BA}$ (Fig. 195*b*) because $\mathbf{v}_{AB}$ and $\mathbf{v}_{BA}$ are perpendicular to one and the same line AB and are in the direction of its rotation about different ends (it is clockwise in both instances).

Constructing a parallelogram on the vectors $\mathbf{v}_B$ and $\mathbf{v}_{AB}$ which are now the components of the velocity $\mathbf{v}_A$ of point A, we can find $\mathbf{v}_A$ as the diagonal of that parallelogram (Fig. 195*b*).

Problem 86. In Fig. 196, the rod AB is in motion in the plane Oxy in a manner such that its end A slides along the x-axis and the rod itself is in contact with a vertical wall OC. Find the velocity of point C of the rod (the point of contact with the wall) and also the angular velocity ω of the rod for the time when the axis of AB is inclined to the x-axis at $\varphi = 60°$ and the velocity of the lower end of the rod is $v_A = 4$ m/s. The wall OC is 2 m high.

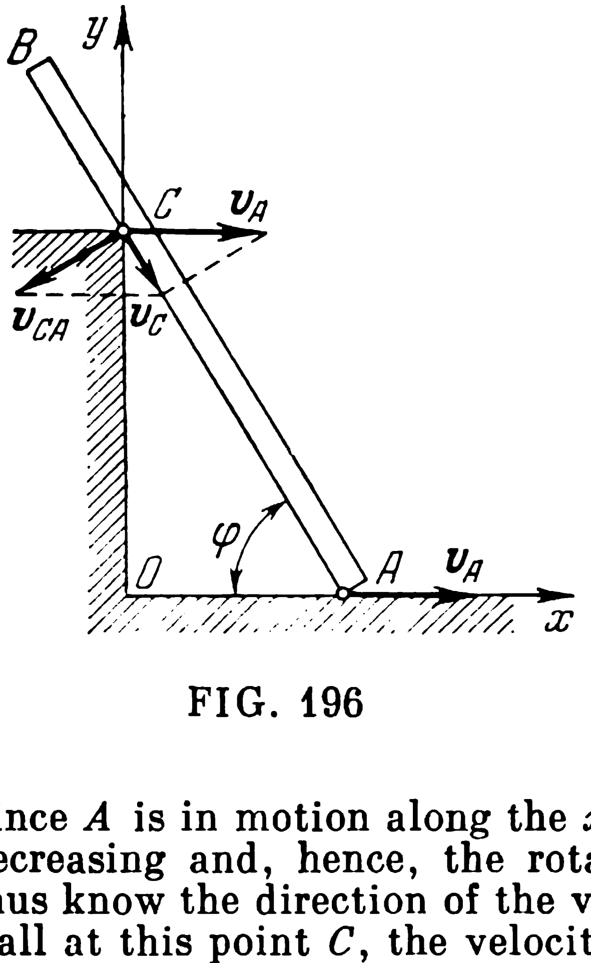


FIG. 196

Solution. The rod AB performs plane motion. We know the velocity of one point, A, and so we take that point for the pole. Then the velocity of the point C of the rod is $\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{CA}$.

The rotational velocity of C about A is equal, in magnitude, to $v_{CA} = \omega \times CA$, where ω is the angular velocity of the rod AB. The velocity $\mathbf{v}_{CA}$ is perpendicular to CA in the direction of rotation. Since A is in motion along the x-axis to the right, the angle φ will be decreasing and, hence, the rotation of AB is counterclockwise. We thus know the direction of the velocity $\mathbf{v}_{CA}$. Since the rod touches the wall at this point C, the velocity $\mathbf{v}_C$ of this point can only* be along the rod. On the other hand, the vector $\mathbf{v}_C$ must be a diagonal of the parallelogram constructed on the vector $\mathbf{v}_A$ (both the magnitude and direction of which we know) and on the vector $\mathbf{v}_{CA}$ (the direction of which is known) as sides. With these conditions we can construct only one parallelogram (Fig. 196). Since the angle between $\mathbf{v}_C$ and $\mathbf{v}_{CA}$ is a right angle, the right triangle yields

$$v_C = v_A \cos \varphi = v_A \cos 60° = 4 \times 0.5 = 2 \text{ m/s}.$$

$$v_{CA} = v_A \sin \varphi = v_A \sin 60° = 4 \frac{\sqrt{3}}{2} = 2\sqrt{3} \text{ m/s}.$$

---

* If the velocity of this point were in a different direction, then it would be resolved into two components: one directed along the rod and the other at right angles to the rod. But the latter component is excluded because the wall prevents motion of C perpendicular to the rod.

From the right triangle OAC we find the length of AC for the given time:

$$AC=\frac{OC}{\sin\varphi}=\frac{OC}{\sin 60^\circ}=\frac{2}{\frac{\sqrt{3}}{2}}=\frac{4}{3}\sqrt{3}\text{ m}.$$

The angular velocity of the rod AB at the given time is

$$\omega=\frac{v_{CA}}{CA}=\frac{2\sqrt{3}}{\frac{4\sqrt{3}}{3}}=1.5\text{ rad/s}.$$

Sec. *75. A theorem on the projections of the velocities of two points of a figure

Theorem. *The projections of the velocities of two points of a figure on the straight line joining the points are equal.*

Proof. Taking some point A of the figure for the pole (Fig. 197), we obtain for the velocity of another arbitrary point B of the figure the [following vector equation [formula (117)]: $\mathbf{v}_B=\mathbf{v}_A+\mathbf{v}_{BA}$. Projecting the vectors that appear in both members of this equation on the straight line AB, we find $(\mathbf{v}_B)_{AB}=(\mathbf{v}_A)_{AB}+(\mathbf{v}_{BA})_{AB}$. But the vector $\mathbf{v}_{BA}$ of rotational velocity of point B about A is perpendicular to AB and therefore its projection on AB is zero, that is, $(\mathbf{v}_{BA})_{AB}=0$. We thus get $(\mathbf{v}_B)_{AB}=(\mathbf{v}_A)_{AB}$ and the proof is complete.

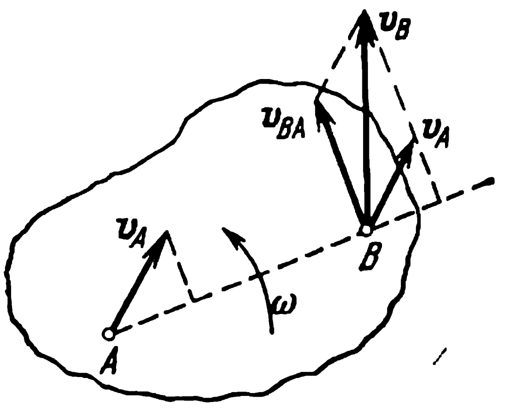


FIG. 197

Problem 87. Referring to Fig. 198, a rectilinear line AB is in motion in a plane, and the velocities of its end points form with AB

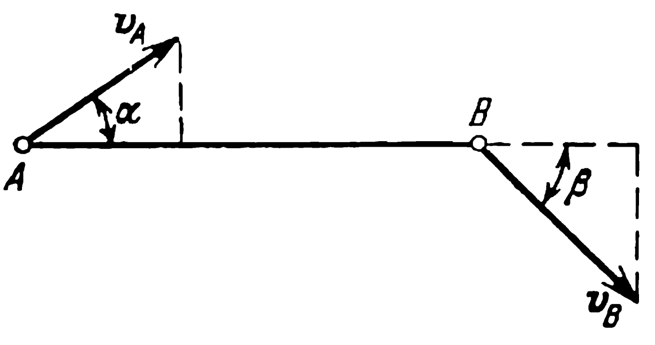


FIG. 198

angles α and β. Find the velocity $\mathbf{v}_B$ if we know the velocity $\mathbf{v}_A$ of point A.

Solution. By the theorem on the projections of velocities of two points of a figure on the straight line joining the points, we have

$$v_A \cos\alpha = v_B \cos\beta,$$

whence

$$v_B = v_A \frac{\cos\alpha}{\cos\beta}.$$

Sec. 76. The instantaneous centre of velocities of a figure

In Sec. 75 we established that the velocity of any point of a plane figure at every given instant of time may be regarded as the geometric sum of two velocities: the velocity of the pole (the pole being any point of the figure) and the rotational velocity of the given point about the pole. This arbitrariness in the choice of pole permits substantially simplifying the study of motion of a plane figure.

For any motion of the figure (with the exception of translational motion) it is always possible to find a point (either on the figure or on an imaginary extension of the figure) whose velocity at the given time is zero. Indeed, suppose at a given time the velocity of an arbitrary point A of the figure (Fig. 199) is $\mathbf{v}_A$ and the direction of rotation of the figure (indicated by an arrow in Fig. 199) and its angular velocity ω are known. Draw a perpendicular at A to the velocity of that point so that a 90° angle is reckoned from the vector $\mathbf{v}_A$ in the direction of rotation of the figure, and on it lay off $AP = v_A/\omega$. Take A for the pole. Then the velocity of P (like the velocity of any other point of a figure* in plane motion) will be a combination of the velocity $\mathbf{v}_A$ of pole A and the rotational velocity $\mathbf{v}_{PA}$ of point P about that pole:

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{PA}.$$

The magnitude of the rotational velocity

$$v_{PA} = \omega \times AP = \omega \frac{v_A}{\omega} = v_A,$$

---

* It may turn out that the instantaneous centre of velocities P lies outside the contour of the given plane figure. If we imagine an unbounded plane attached rigidly to the figure, we can still regard P as belonging to the moving figure.

which is to say it is equal to the magnitude of the velocity $\mathbf{v}_A$ of the pole. But the rotational velocity $\mathbf{v}_{PA}$ is perpendicular to AP in the direction of rotation of the figure, that is, along the same straight line as the velocity $\mathbf{v}_A$ of the pole but in the opposite direction (Fig. 199).

Clearly, the geometric sum of two opposite vectors $\mathbf{v}_A$ and $\mathbf{v}_{PA}$ is zero, and therefore at the given time the absolute velocity of P, that is, its velocity with respect to the fixed plane in which the figure is moving is zero, $\mathbf{v}_P = 0$.

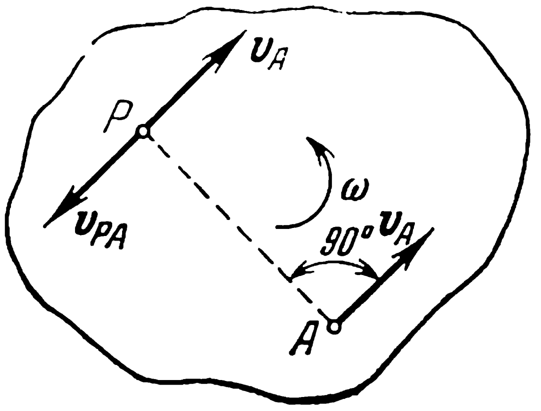


FIG. 199

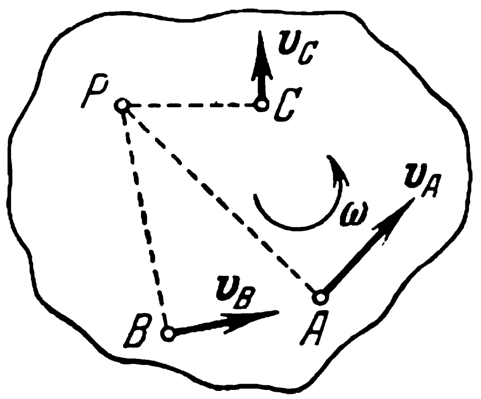


FIG. 200

A point P which is rigidly attached to a moving plane figure and whose velocity at a given time is zero is termed the instantaneous centre of velocities of the figure.

The instantaneous centre P of velocities of a figure, as is evident from the method by which we found its position, always lies on the line drawn from some point of the figure perpendicularly to the direction of the velocity of that point.

If the directions of the velocities of any two points of a figure are known, the *instantaneous centre P of the velocities of the figure is readily found as the point of intersection of lines drawn from the given points of the figure perpendicularly to the velocity vectors of the points* (Fig. 200).

If we find the position of the instantaneous centre P of velocities and if we know, for a given time, the velocity of some point A of the figure not only in direction but also in absolute value, then it is easy to find the angular velocity of the figure corresponding to that time. Since

$$AP = \frac{v_A}{\omega},$$

it follows that

$$\omega = \frac{v_A}{AP}. \tag{118}$$

At every instant of time, the angular velocity of the figure is equal to the ratio of the magnitude of the velocity (corresponding to that time) of some point of the figure to the distance from that point to the instantaneous centre of velocities. The sense of rotation of the figure is determined by the known direction of the velocity of its point.

The foregoing procedure for determining the instantaneous centre of velocities of a figure as the point of intersection of perpendiculars erected to the velocity vectors of two points of the figure is obviously not applicable when the velocities are parallel. Two cases are then possible.

1. *The velocities of two points A and B of a figure are parallel but the points do not lie on a single perpendicular to the direction of the velocities* (Fig. 201).

Since the perpendiculars erected from points A and B to their velocities do not intersect, the instantaneous centre of velocities of the figure does not exist here (it lies at infinity). The distances of the given points from the instantaneous centre of velocities is $AP = BP = \infty$. At the given time, the angular velocity of the figure is

$$\omega = \frac{v_A}{\infty} = \frac{v_B}{\infty} = 0.$$

Hence, the figure is not in rotation at that time. And since every absolute plane motion of a figure may be regarded as a combination of translational motion with the velocity of an arbitrarily chosen pole and rotational motion about that pole (with angular velocity ω, which is independent of the choice of the pole), it follows that the absolute velocities of the points of the figure are then equal merely to the velocity of the pole. In other words, *at the given time the figure is in translational* motion and the velocities of all its points are equal.*

---

* In contrast to continuous translational motion of a body during which the velocities of all its points are equal at every instant of time, the motion of a body in which the velocities of its points are equal only at a given time is sometimes termed instantaneous translational motion.

2. *The velocities of two points A and B of a figure are parallel and the points lie on a single perpendicular to the directions of the given velocities* (Fig. 202).

In this case, the perpendiculars erected at points A and B of the figure to the directions of their velocities merge into one straight line, and to determine the position of the instantaneous centre of velocities we must know the velocities of the two points both in direction and in magnitude.

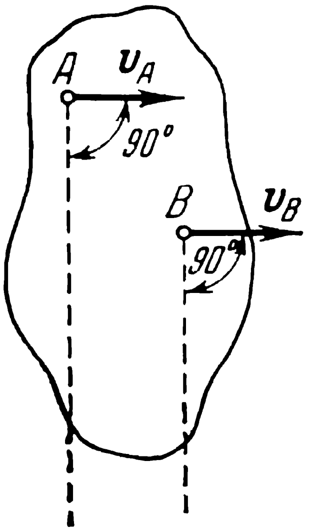


FIG. 201

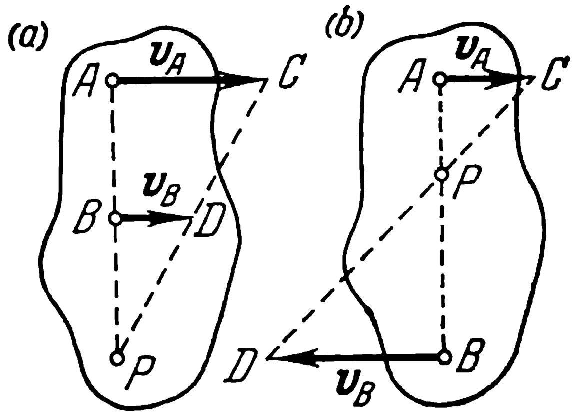


FIG. 202

Since the instantaneous centre of velocities always lies on a perpendicular erected at any point of the figure to the direction of its velocity, and the magnitudes of the velocities of the different points of the figure are, at every instant of time, proportional to the distances of these points from the instantaneous centre, the position P of this point on the perpendicular may be found from the proportion $AP/BP = v_A/v_B$.

If (referring to Figs. 202*a* and 202*b*) we draw a straight line through the end points C and D of vectors $\mathbf{v}_A$ and $\mathbf{v}_B$, then it will intersect the line AB (which joins the given points of the figure) at point P, which is the instantaneous centre of velocities of the figure. Indeed, we can set up the foregoing proportion by proceeding from the similarity of the resulting triangles PAC and PBD.

If in so doing $v_A = v_B$, then the figure is, at the given time, in translational motion (as in the preceding case).

In practical problems, one often has to do with cases where a plane figure is in motion so that its contour rolls along a certain fixed curve without slipping. Since at every given time a moving plane figure can have only one point with velocity zero, and in the case of rolling without slipping that point is a point of the figure at which it is tangent to a fixed curve, it follows that *in the rolling (without slipping) of the contour of the figure along a fixed curve the instantaneous centre of velocities is the point of contact of the contour with the fixed curve.*

Sec. 77. Distribution of velocities of points of a plane figure

Taking the instantaneous centre P of velocities of a figure as the pole, we can readily find the velocities of all the other points of the figure at that time:

$$\begin{gathered}\mathbf{v}_A=\mathbf{v}_P+\mathbf{v}_{AP}=\mathbf{v}_{AP}, \quad \mathbf{v}_B=\mathbf{v}_P+\mathbf{v}_{BP}=\mathbf{v}_{BP}, \\ \mathbf{v}_C=\mathbf{v}_P+\mathbf{v}_{CP}=\mathbf{v}_{CP}\end{gathered} \tag{119}$$

and so on.

Hence, *the velocity of any point of a plane figure is equal to the rotational velocity of that point about the instantaneous centre of velocities of the figure.*

On this basis it is easy to find the magnitude and direction of velocity of each point of the figure (if the position of the instantaneous centre P and the angular velocity of the figure, which is independent of the choice of pole, are known at a given time): $v_A = \omega \times AP$, $v_B = \omega \times BP$ and so forth [formula (119)]. Dividing both sides of these equalities termwise, we get $v_B/v_A = BP/AP$.

The magnitudes of the velocities of distinct points of a figure are at every given instant proportional to the distances of the points from the instantaneous centre of velocities of the figure corresponding to the given instant. The directions of the velocities of different points of the figure are perpendicular to the lines joining the corresponding points with the instantaneous centre of velocities and are in the direction of rotation of the figure (Fig. 200).

Thus, *at every instant the velocities of distinct points of a plane figure are distributed as if the figure were in rotation, at that time, about an instantaneous centre of velocities* that occupies distinct positions at distinct times both with respect to the moving figure and with respect to the fixed plane in which the figure moves.

If we do not consider the motion of a plane figure obtained by cutting a body with a plane parallel to a fixed plane, but rather the plane-parallel motion of the body itself, then the velocities of its points will, at every instant of time, be distributed as if the body were in rotation, at that time, about a certain so-called *instantaneous axis of rotation*, which passes through the instantaneous centre of velocities of the figure at that time and is perpendicular to its plane. This follows from the fact that in the plane motion of a body all points lying on one perpendicular to the fixed plane move in the same manner. For that reason, all points of the body that lie on the instantaneous axis of rotation, that is, on the straight line passing through the instantaneous centre of velocities of the figure and perpendicular to its plane will, at the given time, have a velocity equal to zero, and all points of the body that lie on the perpendicular (erected at some other point) to the plane of the figure will have the same velocities as the corresponding point of the figure.

To each time (instant) *there corresponds a specific position of the instantaneous centre of velocities and a specific position of the instantaneous axis.* This is indicated by the terms themselves: instantaneous centre and instantaneous axis.

Note that *it is impossible to identify the rotation of a body about an instantaneous axis at a given time and the rotation of the body about a fixed axis.*

In the latter case, the velocities of all points of the body that lie on the axis are zero as long as the body is in motion and so their accelerations are zero. Now the points of the body that coincide at the given time with the instantaneous axis of rotation have velocities equal to zero only at that time and are in accelerated motion.

Problem 88. Referring to Fig. 203, a cylinder lying on a horizontal plane is wound with a rope, one end of which is attached to the cylinder and the other is free. Find the angular velocity ω of the cylinder, the velocity of its centre O and the velocities of the end points of the vertical and horizontal diameters of a perpendicular section of the cylinder: A_1, A_2, A_3 and A_4 if the free end of the rope is being drawn parallel to the plane and perpendicularly to the axis of the cylinder with a constant velocity v_3 and if the cylinder rolls without sliding.

Solution. Pass a plane through the rope at right angles to the axis of the cylinder. In the section we obtain a circle rolling along a horizontal straight line, the line of intersection of the drawn plane and the horizontal plane. Since this circle rolls without slipping, the velocity of point A_1 at which it is tangent to the straight line is zero. This point is thus the instantaneous centre of velocities of the circle. The velocity of point A_3 of the circle at which the free end of the rope is in contact is equal to the velocity v_3 with which the rope is being drawn. Knowing the position of the instantaneous centre of velocities of the circle and the velocity of the single point A_3, we can find the angular velocity of the cylinder and the velocities of the points O, A_2 and A_4:

$$\omega=\frac{v_3}{A_1A_3}=\frac{v_3}{2r}, \qquad v_0=\omega\times A_1O=\frac{v_3}{2r}\,r=\frac{v_3}{2},$$

$$v_2=\omega\times A_1A_2=\frac{v_3}{2r}\sqrt{2r^2}=\frac{v\sqrt{2}}{2}, \quad v_4=\omega\times A_1A_4=\frac{v_3\sqrt{2}}{2}.$$

The directions of the vectors $\mathbf{v}_0$, $\mathbf{v}_2$ and $\mathbf{v}_4$ are shown in Fig. 203.

Problem 89. Referring to Fig. 204*a*, find the velocity of slide block B and the midpoint M of connecting rod AB of the crank-and-slide mechanism at the time when the crank OA forms any angle φ with the line OB of motion of the slide block. Compute the velocities of these points (B and M) for two positions of the mechanism: when $\varphi = 0$ and $\varphi = 90°$ on the assumption that $OA = r = 20$ cm, $AB = l = 40$ cm. The angular velocity of the crank is constant and equal to $\omega = 40\pi$ rad/s. The direction of its rotation is shown in the drawing.

Solution. At all times we know the directions of the velocities of two points A and B of the connecting rod. A is a point of the rod AB that is common to the crank OA as well. But the crank is in rotational motion and therefore its point A moves in a circle of radius OA and, hence, the velocity $\mathbf{v}_A$ of this point is perpendicular to the radius OA.

Point B is a point of the connecting rod that is common to the slide block B too which can only move translationally along its guideways and, hence, the velocity of B is along the line of motion OB of the rod. Now, if we know the directions of the velocities of two points of the connecting rod for a given time, then it is also easy to find the position of its instantaneous centre of velocities. This point P lies at the intersection of perpendiculars erected at A and B of the connecting rod to the directions of the velocities of these points (Fig. 204*a*). The angular velocity of the rod is $\omega_{AB} = v_A/AP$ and the magnitude of the velocity of point M of the rod is

$$v_M=\omega_{AB}\times MP=\frac{v_A\times MP}{AP}.$$

The magnitude of the velocity of the slide block (point B of the rod) is

$$v_B=\omega_{AB}\times BP=\frac{v_A\times BP}{AP}.$$

The velocities of these points are perpendicular respectively to lines *MP* and *BP* in the direction of rotation of the connecting rod (as shown in Fig. 204*a*). The velocity of *A* can easily be found from the formula $v_A = \omega \times OA = \omega r$. Knowing the angle φ for a given time, it is always possible to find the lengths of *AP*, *MP* and *BP* either by computation (solving the appropriate triangles) or graphically (by constructing a scheme of the mechanism to scale on the basis of the given dimensions of its members and the angle φ).

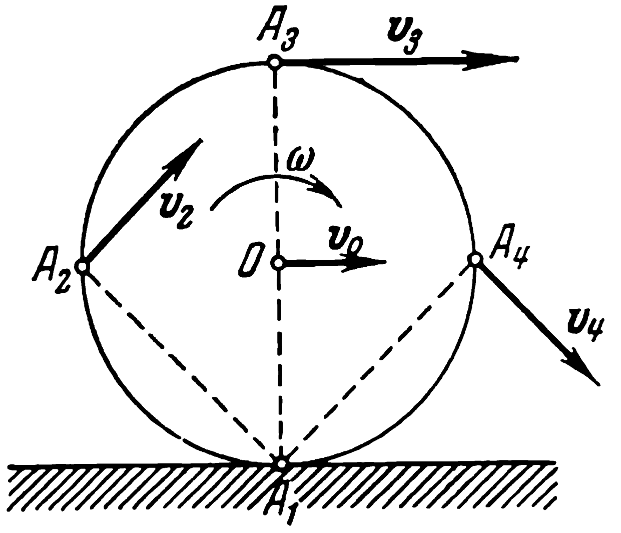


FIG. 203

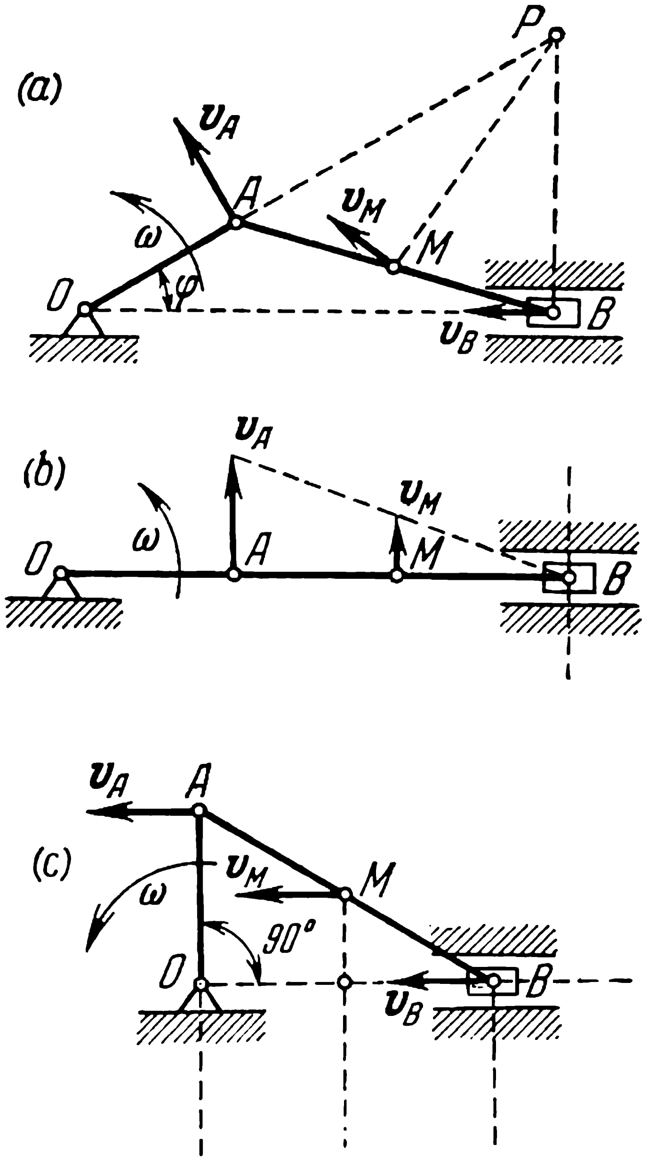


FIG. 204

Now let us solve this same problem for the special cases of $\varphi = 0$ and $\varphi = 90°$. The arrangement of the members of the mechanism and the direction of the velocity $\mathbf{v}_A$ of point *A* for the angle of rotation of the crank $\varphi = 0$ is shown in Fig. 204*b*. The velocity of point *B* in all cases may be only along the line of motion of the slide block. The instantaneous centre of the connecting rod *AB* is *B* since it is here, in the given case, that the perpendiculars erected at points *A* and *B* to the velocities of these points intersect. Since the point *B* is located in the instantaneous centre at the given time, the velocity of this point (the velocity of the slide block) is zero (the mechanism is in the dead centre position). The angular velocity of the connecting rod is

$$\omega_{AB} = \frac{v_A}{AP} = \frac{v_A}{AB},$$

The velocity of point M of the connecting rod is

$$v_M=\omega_{AB}\times MP=\omega_{AB}\times MB=\frac{v_A\times MB}{AB}=\frac{\omega r\frac{1}{2}}{l}$$

$$=\frac{\omega r}{2}=\frac{40\pi\times 20}{2}=400\pi \text{ cm/s}.$$

The arrangement of the mechanism and the directions of the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ for an angle of rotation of the crank $\varphi = 90°$ is shown in Fig. 204*c*. Since the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are parallel and the points A and B do not lie on one perpendicular to the directions of these velocities, it follows that at the given time the instantaneous centre of velocities of the connecting rod AB lies at infinity, its angular velocity $\omega_{AB} = 0$ and it is executing instantaneous translational motion. Hence, at the given time,

$$v_B = v_M = v_A = \omega r = 40\pi \times 20 = 800\pi \text{ cm/s}.$$

Problem 90. Two parallel racks are moving in different directions with constant velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ (Fig. 205). Held between the racks is a disc of radius r rolling along the racks without sliding. Find the angular velocity of the disc and the velocity of its centre O if $v_1 > v_2$.

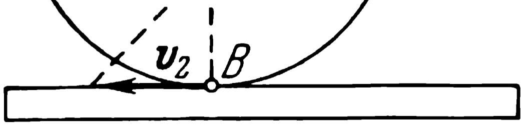


FIG. 205

Solution. Since the disc rolls along the racks without sliding, the velocity of its point A at which the disc touches the upper rack is equal to the velocity of this rack, $\mathbf{v}_1$. So also the velocity of point B of the disc is equal to the velocity of the lower rack, $\mathbf{v}_2$. Since the velocities of the two points A and B of the disc are parallel, and the points themselves lie on a single perpendicular to the velocities of the points, the instantaneous centre P of velocities of the disc is defined as the point of intersection of the straight line AB and the straight line joining the end points of the velocity vectors of A and B.

From the similarity of the triangles (this also follows from the proportionality of the velocities of the points of the plane figure to their distances from the instantaneous centre), we have $AP/BP = v_1/v_2$. On the other hand, $AP + BP = AB = 2r$. Solving these two equations simultaneously, we get

$$BP=\frac{2rv_2}{v_1+v_2}.$$

The angular velocity of the disc is $\omega = \frac{v_2}{BP} = v_2 \div \frac{2rv_2}{v_1 + v_2} = \frac{v_1 + v_2}{2r}$. To determine the velocity of the centre O of the disc, we first compute the distance of this point from the instantaneous centre P:

$$OP = OB - BP = r - \frac{2rv_2}{v_1 + v_2} = \frac{r(v_1 - v_2)}{v_1 + v_2}.$$

The velocity of the centre O of the disc is

$$v_O = \omega \times OP = \frac{v_1 + v_2}{2r} \frac{r(v_1 - v_2)}{v_1 + v_2} = \frac{v_1 - v_2}{2}.$$

Sec. *78. Composition of rotations about parallel axes

Suppose a rigid body K (say, a ball, as shown in Fig. 206*a*) is in rotation about an axis z_1 (relative rotation), which in turn is in rotation about a fixed axis z_2 parallel to z_1 (migratory rotation).

It is obvious that here all points of the body will be moving in planes perpendicular to parallel axes z_1 and z_2 and therefore any of these planes may be regarded as fixed, parallel to which all the points of the body are in motion. Thus, *the absolute motion of a body participating in two rotations about parallel axes is a special case of plane-parallel motion of the body*, and to determine such motion it suffices to consider the motion of a plane figure S (Fig. 206*b*) which constitutes a section of the body by a plane perpendicular to the given axes.

Now, the motion of a plane figure in its plane may, as we know, be regarded at each given time as a rotational motion of the figure about the instantaneous centre (corresponding to this time) with a certain absolute angular velocity ω. When determining the position of the instantaneous centre of velocities of the figure and its absolute angular velocity, three cases may occur. We will consider each of them.

First Case. Both rotations have the same sense (Fig. 206*a*). Denote the traces of the axes z_1 and z_2 in the plane of figure S by the points O_1 and O_2*. The point O_1 lies on the axis

---

* The point O_2 may even lie outside the outline of the given figure S, in which case it must be mentally attached rigidly to the figure.

z_1 and therefore has only the velocity due to rotation about the axis z_2. Its velocity is $v_1 = \omega_2 \times O_1O_2$. The point O_2 lies on the extension of axis z_2 and has only the velocity due to rotation about the axis z_1. Its velocity is $v_2 = \omega_1 \times O_1O_2$.

We have established that the velocities of the two points of figure S are parallel, while the points themselves lie on

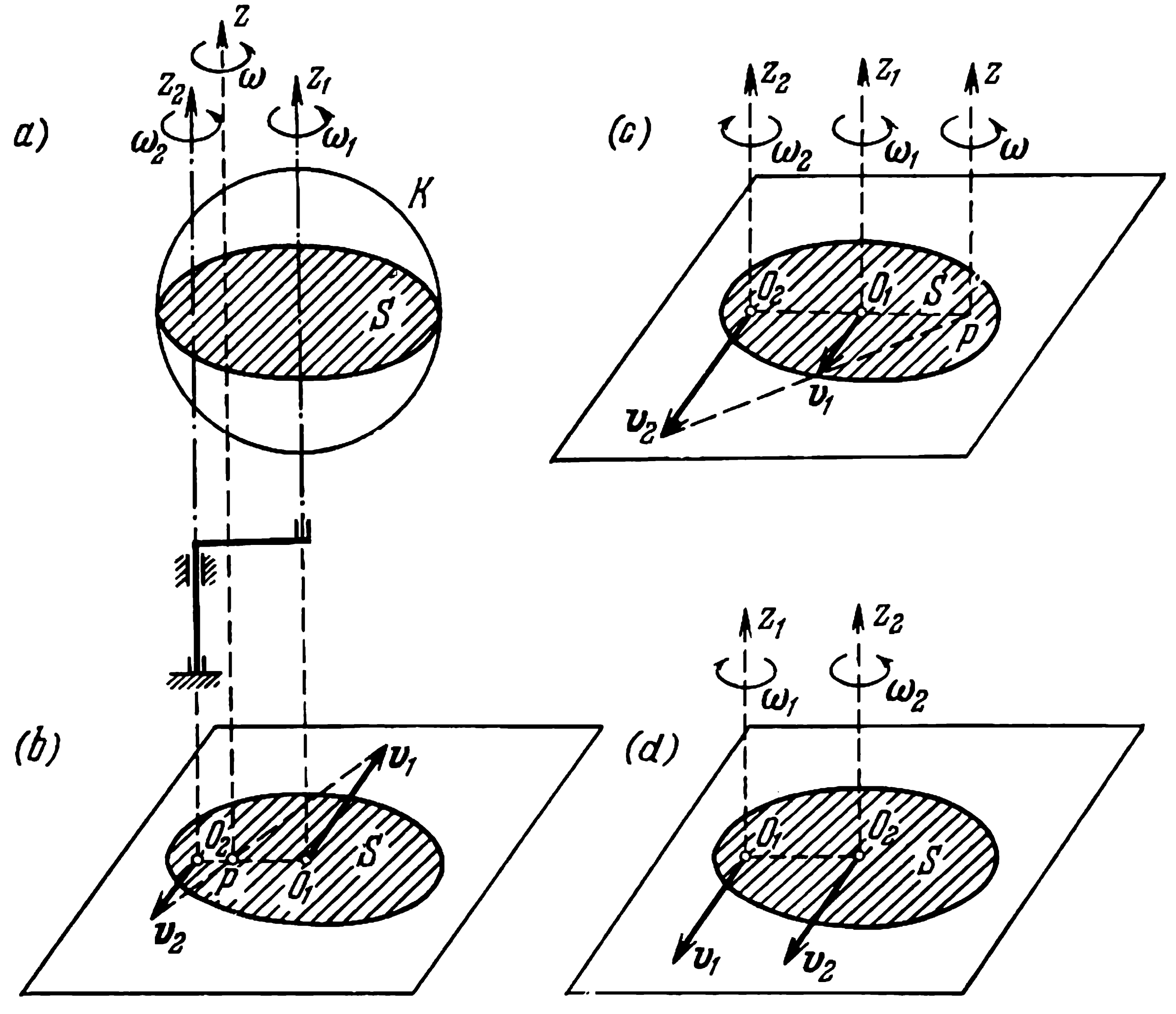


FIG. 206

a single perpendicular to the direction of the velocities (Fig. 206*b*). In accordance with what was said in Sec. 76.2, the instantaneous centre P of the velocities of the figure then lies on the straight line O_1O_2 and is separated from points O_1 and O_2 by distances that are directly proportional to the velocities of these points:

$$\frac{PO_1}{PO_2} = \frac{v_1}{v_2} = \frac{\omega_2 \times O_1O_2}{\omega_1 \times O_1O_2} = \frac{\omega_2}{\omega_1}.$$

Since in plane motion, all points of a body lying on one perpendicular to a fixed plane move in the same way, so the velocities of all points of the straight line z, which is parallel to the given axes and passes through the instantaneous centre P thus found, are zero. For the given body K, this straight line is the instantaneous axis, rotation about which at a given time is the absolute motion of that body. Now let us determine the angular velocity ω of rotation of the body about the instantaneous axis, that is, the absolute angular velocity of the body.

By formula (118),

$$\omega = \frac{v_2}{PO_2} = \frac{v_1}{PO_1} \quad \text{or} \quad \omega = \frac{\omega_1 \times O_1O_2}{PO_2} = \frac{\omega_2 \times O_1O_2}{PO_1}.$$

Whence we get $\omega \times PO_2 = \omega_1 \times O_1O_2$ and $\omega \times PO_1 = \omega_2 \times O_1O_2$. Adding both sides of these equations termwise, we obtain

$$\omega_1 \times O_1O_2 + \omega_2 \times O_1O_2 = \omega (PO_2 + PO_1) = \omega \times O_1O_2,$$

whence

$$\omega = \omega_1 + \omega_2.$$

Since the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of points O_1 and O_2 in rotation about the instantaneous centre P must clearly be directed in the same way as in rotation about the axes z_2 and z_1, the rotation of the body about the instantaneous axis must be in the same direction (Fig. 206).

We thus arrive at the following conclusion.

(1) *Two rotations that take place about parallel axes in the same direction may at each instant of time be replaced by a single rotation in the same direction about the instantaneous axis (parallel to the given axes) that lies in the same plane and divides the distance between them into parts that are inversely proportional to the angular velocities of the component rotations*:

$$\frac{PO_2}{PO_1} = \frac{\omega_1}{\omega_2}. \tag{120}$$

(2) *The absolute angular velocity of a body is equal to the sum of the angular velocities of the component rotations*:

$$\omega = \omega_1 + \omega_2. \tag{121}$$

If the ratio of the angular velocities $\omega_1/\omega_2 = PO_2/PO_1$ is constant and if the distance O_1O_2 is too, then the instantaneous axis of rotation will be at an invariable distance PO_2 from the fixed axis z_2. But as the body moves, the line O_1O_2 rotates about the fixed centre O_2 and therefore the instantaneous axis of rotation is constantly changing its position in space.

Second Case. The two rotations are in different directions and with different angular velocities. Imagine, as in the preceding case, a plane perpendicular to the given parallel axes, and in motion in it imagine a plane figure S (Fig. 206*c*), which is a section of the body by that plane. Suppose the body is in rotation about axis z_1 that passes through point O_1 of the figure and the rotation is counterclockwise if one looks from the positive z_1-axis with angular velocity ω_1, while the rotation of the body about the z_2-axis, which passes through O_2, is clockwise with angular velocity ω_2. Also let $\omega_1 > \omega_2$.

Reasoning as in the preceding case, we find that the velocity of point O_1 is equal to $v_1 = \omega_2 \times O_1O_2$ and the velocity of O_2 is $v_2 = \omega_1 \times O_1O_2$. The velocities of these points are also parallel, but this time in the same direction (Fig. 206*c*). The instantaneous centre P of the velocities of the figure lies in this case on the extension of the line O_1O_2 beyond the axis with the greater angular velocity and is separated from points O_1 and O_2 by distances that are proportional to the velocities of the points:

$$\frac{PO_1}{PO_2} = \frac{v_1}{v_2} = \frac{\omega_2 \times O_1O_2}{\omega_1 \times O_1O_2} = \frac{\omega_2}{\omega_1}.$$

The straight line z, which is parallel to the given axes and passes through the instantaneous centre P of velocities is the instantaneous axis of rotation.

To determine the absolute angular velocity ω, that is, the angular velocity of rotation of the body about the instantaneous axis z, we take advantage of formula (118):

$$\omega = \frac{v_2}{PO_2} = \frac{v_1}{PO_1} \quad \text{or} \quad \omega = \frac{\omega_1 \times O_1O_2}{PO_2} = \frac{\omega_2 \times O_1O_2}{PO_1}.$$

From this we find $\omega \times PO_2 = \omega_1 \times O_1O_2$ and $\omega \times PO_1 = \omega_2 \times O_1O_2$. Subtracting the second equation from the first termwise, we obtain

$$\omega (PO_2 - PO_1) = \omega_1 \times O_1O_2 - \omega_2 \times O_1O_2$$

or

$$\omega \times O_1O_2 = (\omega_1 - \omega_2) \times O_1O_2,$$

whence $\omega = \omega_1 - \omega_2$.

Since in the rotation of the body about the instantaneous axis z the directions of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of points O_1 and O_2 must be the same as in the rotation of these points about the axes z_2 and z_1, the rotation of the body about the instantaneous axis must be (as is evident from Fig. 206*c*) in the direction of rotation of the greater angular velocity.

We thus arrive at the following conclusion.

(1) *Two rotations about parallel axes in different directions with different angular velocities may be replaced at each given instant of time by a single rotation about the instantaneous axis, which is parallel to the given axes, in the direction of the greater angular velocity. The instantaneous axis of a compound rotation lies in the same plane as the given axes beyond the axis of rotation with the greater angular velocity and is separated from the given axes by distances that are inversely proportional to the angular velocities of the component rotations*:

$$\frac{PO_2}{PO_1} = \frac{\omega_1}{\omega_2}. \tag{122}$$

(2) *The absolute angular velocity of the body is equal here to the difference between the angular velocities of the component rotations*:

$$\omega = \omega_1 - \omega_2. \tag{123}$$

Third Case. The two rotations are in different directions and with angular velocities equal in numerical value. Above we found that when combining two rotations about parallel axes that occur in opposite directions with different angular velocities ω_1 and ω_2, the position of the instantaneous axis of rotation and the absolute angular velocity ω are given by the formulas (122) and (123):

$$\frac{PO_2}{PO_1}=\frac{\omega_1}{\omega_2}$$

and

$$\omega=\omega_1-\omega_2.$$

Subtracting unity from both sides of the first equation, we get

$$\frac{PO_2}{PO_1}-1=\frac{\omega_1}{\omega_2}-1,$$

whence

$$\frac{PO_2-PO_1}{PO_1}=\frac{\omega_1-\omega_2}{\omega_2}$$

and

$$\frac{O_2O_1}{PO_1}=\frac{\omega_1-\omega_2}{\omega_2}.$$

Hence the distance of the instantaneous axis of rotation (passing through the instantaneous centre P) from the axis of relative rotation (passing through point O_1) is

$$PO_1=\frac{\omega_2}{\omega_1-\omega_2}\times O_1O_2.$$

Imagine that the angular velocity ω_2 of migratory rotation approaches without bound the angular velocity ω_1 of relative rotation. Then in the limit, as $\omega_2\rightarrow\omega_1$, we have for the absolute angular velocity of the body

$$\omega=\lim_{\omega_2\rightarrow\omega_1}(\omega_1-\omega_2)=0.$$

The distance of the instantaneous axis from the axis of relative rotation is

$$PO_1=\lim_{\omega_2\rightarrow\omega_1}\frac{O_2O_1\times\omega_2}{\omega_1-\omega_2}=\infty.$$

In Sec. 77 it was established that in this case the plane motion of the body is translational motion and, hence, the velocities of all its points at every instant of time are equal. This may be illustrated in a different way.

Again imagine a plane perpendicular to the given parallel axes, and moving in it imagine a plane figure (Fig. 206*d*), which is a section of the body by that plane. Suppose the rotations of the body about the axis z_1 passing through point O_1 and about z_2 passing through O_2 are in different directions and the angular velocities of these rotations are equal in numerical value:

$$\omega_1 = \omega_2 = \omega.$$

Reasoning as we did in the preceding cases, we can find that the velocities of points O_1 and O_2 are parallel, in the same direction (Fig. 206*d*), and are equal in magnitude:

$$v_1 = \omega \times O_1O_2 \quad \text{and} \quad v_2 = \omega \times O_1O_2.$$

Now, if two points of a plane figure have the same velocities (in magnitude and direction), the motion of the body at that time will be translational.* All other points of the body will have those velocities.

The velocity of translational motion of a body is equal to the velocity of any one of its points, that is, in this case it is equal to the product of the angular velocity by the distance O_1O_2 between the parallel axes.

We thus arrive at the following conclusion.

(1) *Two rotations of a body about parallel axes in different directions with equal angular velocities may be replaced, at a given time, by a single translational motion occurring in the direction perpendicular to the plane drawn through the axes of the component rotations.*

(2) *Then the magnitude of the velocity of the translational motion of the body is equal at the given time to the product of the angular velocity* ω *of one of the rotations into the shortest distance* O_2O_1 *between the axes of the component rotations*:

$$v = \omega \times O_2O_1. \tag{124}$$

This special case of the composition of two rotations of a body is clearly analogous to the special case of two parallel forces equal in magnitude and in different directions, that is, the case of a force couple. By analogy with the latter, the case of the composition of two rotations occurring about parallel axes in different directions with equal (in magnitude) angular velocities is termed a *rotation couple*. A rotation couple cannot be replaced by a single rotational motion.*

---

* Instantaneous translational if the equality between the component angular velocities is not maintained or if the direction of one of the rotations is reversed.

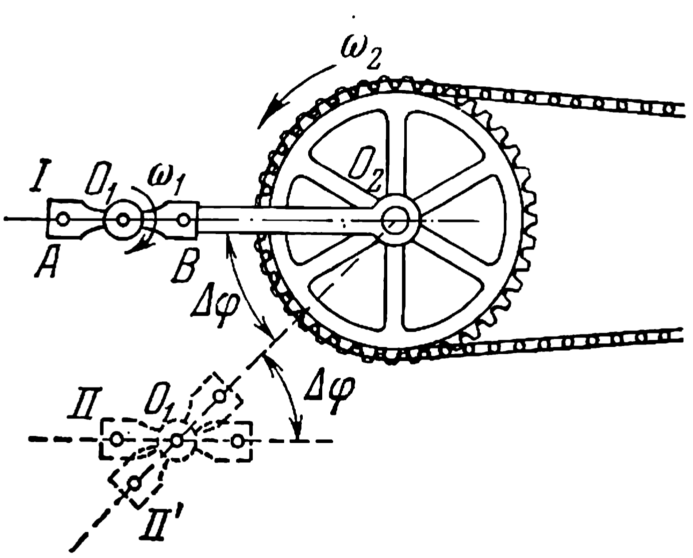


FIG. 207

A simple instance of a rotation couple that clearly illustrates its property is the motion of a bicycle pedal. The pedal *AB* (Fig. 207) is held in the horizontal position by the rider's foot and, relative to the frame of the bicycle, executes translational motion. At the same time, this motion may be regarded as the result of the composition of two rotations occurring about parallel axes in different directions with equal angular velocities: a relative rotation of the pedal about the axis O_1 and a migratory rotation together with crank O_2O_1 about the axis O_2, which is fixed relative to the frame of the bicycle. True enough, suppose for a certain time interval Δt the crank O_2O_1 rotates (as indicated in Fig. 207) through the angle $\Delta\varphi$ from position *I* to position *II*. If the pedal were rigidly attached to the crank, it would occupy position *II'* (its line *AB* would lie on the extension O_2O_1 of the crank). But the existence of axis O_1 enables the pedal to remain horizontal in its new position as well, having turned about the axis O_1 in the opposite direction to the rotation of the crank through an angle equal (they being alternate angles) to $\Delta\varphi$. Since in any displacement of the pedal, the angle of rotation of the crank about axis O_2 is equal to the angle of rotation of the pedal about axis O_1, it follows that the migratory angular velocity ω_2 of the pedal is equal in magnitude, at any instant, to its relative angular velocity ω_1, while their directions are opposite.

---

* Just as the rotational action of a force couple is characterized by its moment, so also the velocity of translational motion of a body which is the result of a rotation couple is defined by the moment of the couple.

Sec. *79. The angular velocity as a vector. The analogy between the composition of angular velocities and the composition of forces

The investigation of compound rotational motion of a body is appreciably simplified if we agree to regard its angular velocity as a vector quantity associated with the axis of rotation of the body.

The vector ω *of angular velocity is a vector directed along the axis of rotation such that when viewed from the end of the vector, one sees the body performing counterclockwise rotation* (Fig. 208). The magnitude of this vector is equal to the absolute value of the angular velocity of the body, $\omega = |d\varphi/dt|$. *The vector* ω *is a sliding vector*, that is, for its point of application we can take any point on the axis of rotation of the body.

Specifying the vector ω of angular velocity of a body fully determines the rotational motion of the body since it permits indicating the position of the axis of rotation of the body, the sense of rotation, and the numerical value of the angular velocity. Justification for regarding the angular velocity as a vector quantity is seen (indirectly of course) in the total analogy between the rules (proved in the preceding section) of the composition of rotations about parallel axes and the corresponding rules of the composition of parallel forces.

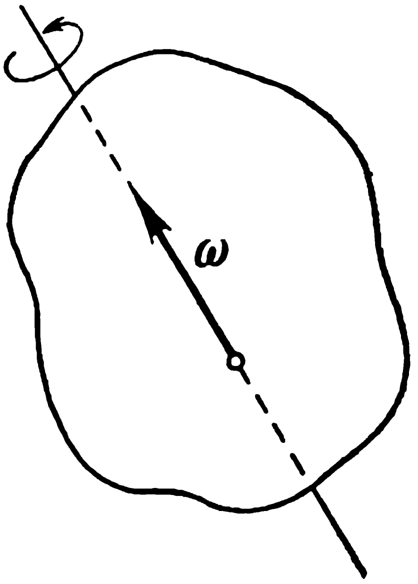


FIG. 208

Suppose a body K is in rotation about an axis z_1 with a certain angular velocity ω_1, and suppose axis z_1 in turn is in rotation about a parallel fixed axis z_2 (Fig. 206a) with a certain angular velocity ω_2.

Let us take some kind of scale for the angular velocities and, using the rule formulated above, let us depict the angular velocities in the form of vectors ω_1 and ω_2 directing them along the appropriate axes of rotation from, say, points O_1 and O_2 (as shown in Fig. 209*a* for the case where the rotations are in the same direction, and Fig. 209*b* for the case where the rotations are in opposite directions and the angular velocities are not equal in absolute value).

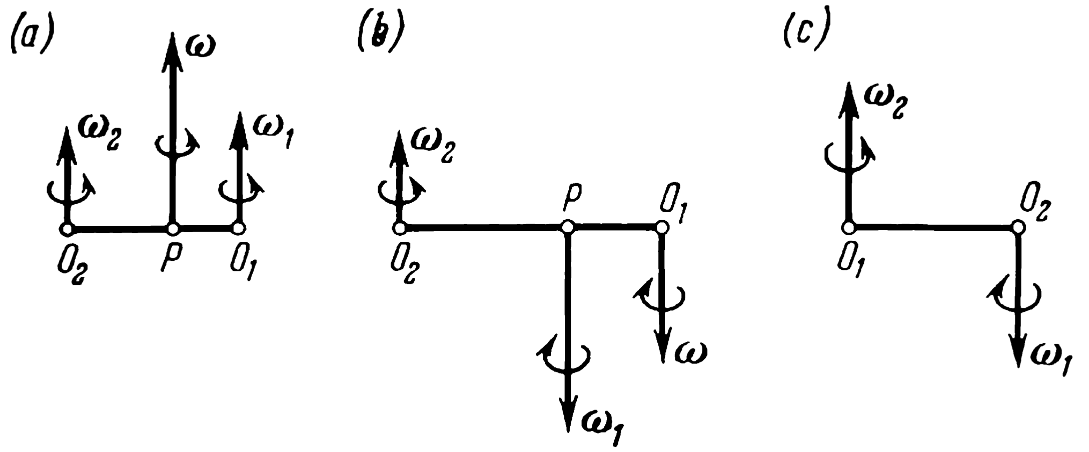


FIG. 209

In both cases shown in Fig. 209, the two rotations of the body about parallel axes may be replaced by a single rotation with an angular velocity whose magnitude, line of action and direction of the vector ω are given by rules similar to the rules established in statics for the composition of parallel forces in the same and in opposite directions (see Secs. 18 and 19).

1. If both rotations are in the same direction,

$$\omega = \omega_1 + \omega_2 \quad \text{and} \quad \frac{PO_2}{PO_1} = \frac{\omega_1}{\omega_2}.$$

2. If the two rotations are in opposite directions and their angular velocities are not equal,

$$\omega = \omega_1 - \omega_2 \quad \text{and} \quad \frac{PO_2}{PO_1} = \frac{\omega_1}{\omega_2}.$$

(The vector ω lies beyond the axis of rotation with the greater angular velocity and is in the direction of the greater angular velocity.)

If the two rotations are in opposite directions and their angular velocities are equal in magnitude ($\omega_1 = \omega_2 = \omega$), as in Fig. 209*c*, then they cannot be replaced by a single rotation (just as in the case of a force couple, that is, two parallel forces equal in magnitude and opposite in direction cannot be replaced by a single resultant force). Here we have the special case known by analogy as a *rotation couple.*

As was demonstrated in the preceding section, a rotation couple imparts translational motion to a body in a direction perpendicular to the plane drawn through the axes of the components of rotation. The magnitude of the velocity of this translational motion is $v = \omega \times O_1O_2$.

Sec. 80. Planetary gear trains and differential gears

Planetary gear trains and differential gears are mechanisms in which there are wheels with moving axes rotating together with a so-called carrier (*H* in Fig. 210) *about a fixed axis.*

The central gears are those whose geometric axes are fixed.

Wheels with moving geometric axes are termed planet wheels.

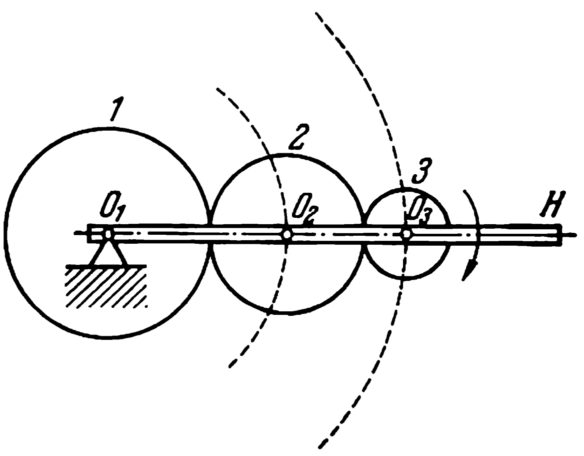


FIG. 210

Wheels with moving axes (*2* and *3* in Fig. 210) execute compound motion, simultaneously rotating about their own axes (O_2 and O_3) fixed on the carrier, and, together with the carrier, revolve about its fixed axis O_1. The motion of these wheels is much like that of planets revolving about the sun in the solar system, whence the name planet wheels, while the mechanism itself is called a planetary gear train. For the same reasons, the central gear is sometimes called a *sun wheel.*

Planetary mechanisms in which one of the central gears is fixed are said to be a simple planetary gear train, or, simply, a planetary gear train.

Unlike these simple planetary gear trains, *planetary mechanisms in which there are no fixed gears are termed differential gears, or simply differentials.*

In differentials, one of the central gears rotates about its fixed axis *independently* of the rotation of the carrier, in other words it obtains its rotation from a different source. Such drives are said to have *two degrees of freedom.*

Planetary and differential gears are widely used in engineering in that they yield large transmission ratios with a small number of gears and (in differentials) the composition of two independent angular velocities.

The kinematic analysis of such gears may be carried out in a variety of ways that are illustrated below in the solutions of problems 91 and 92.

Problem 91. In a planetary gear, the kinematic diagram of which is shown in Fig. 211, gear *1* is fixed, while gear *2* is freely mounted on axis *II* attached to carrier *H*. The carrier is in rotation with angular velocity $\omega_H = 15$ rad/s about fixed axis *I* of the driving shaft, which coincides with the geometric axis *III* of gears *1* and *3*. Rolling along the fixed gear *1* (*1* and *2* are linked by internal gearing), gear *2* drives gear *3*, which is rigidly mounted on the driven shaft *III*. The numbers of teeth in the gears are: $z_1 = 70$, $z_3 = 30$. Find the absolute angular velocity of gear *3* (of the driven shaft *III*).

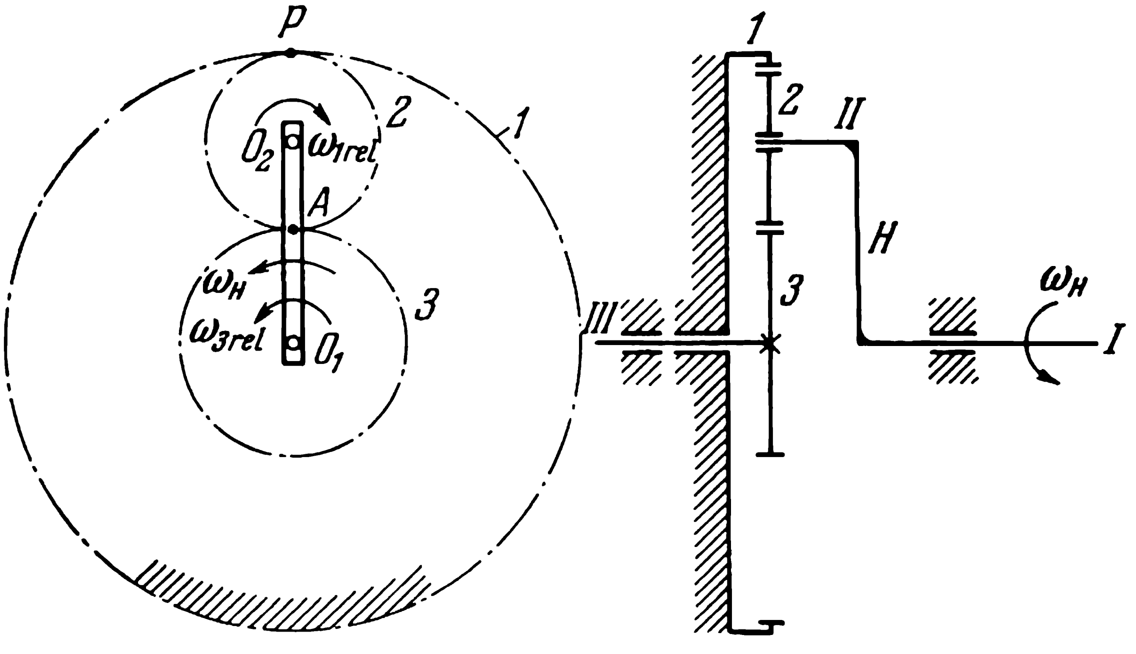


FIG. 211

Solution.

First method (with the aid of the instantaneous centres of velocities). Denote the radii of the pitch circles of the gears by r_1, r_2 and r_3, respectively. Since axis *II* (mounted on carrier *H*) of gear *2* is in rotational motion about fixed axis *I* of the carrier, the velocity of points lying on this axis is

$$v_{O_2} = \omega_H \times O_1O_2 = \omega_H (r_2 + r_3).$$

On the other hand, gear *2* executes plane motion as it rolls along the pitch circle of fixed gear *1*. The instantaneous centre of velocities of gear *2* lies at point P—the point of its contact with the pitch circle of the fixed gear *1*. The velocities of distinct points of gear *2* are proportional to their distances from the instantaneous centre of velocities. Consequently, the velocity v_A of point A (the common point of gears *2* and *3*) is found from the proportion $\frac{v_A}{v_{O_2}} = \frac{AP}{O_2P} = \frac{2r_2}{r_2}$, whence $v_A = 2v_{O_2} = 2\omega_H (r_2 + r_3)$. The absolute angular velocity ω_3 of gear *3* (of the driven shaft) is

$$\omega_3 = \frac{v_A}{r_3} = \omega_H \frac{2(r_2+r_3)}{r_3} = \omega_H \frac{r_3+(r_3+2r_2)}{r_3}$$

$$= \omega_H \frac{r_3+r_1}{r_3} = \omega_H \left(1+\frac{r_1}{r_3}\right).$$

Since the radii of the pitch circles of the gears in mesh are proportional to the numbers of their teeth, we finally get

$$\omega_3 = \omega_H \left(1+\frac{r_1}{r_3}\right) = \omega_H \left(1+\frac{z_1}{z_3}\right) = 15\left(1+\frac{70}{30}\right) = 50 \text{ rad/s}.$$

Second method (the method of composition of angular velocities). The absolute motion of gear *3* may be regarded as composed of two rotations: the rotation of the gear relative to the movable carrier H with relative angular velocity $\omega_{3\,\mathrm{rel}}$ and its rotation together with the carrier about the fixed axis with a migratory angular velocity $\omega_{3\,\mathrm{mig}} = \omega_H$.

The relationship between the relative angular velocities of the gears (that is, between the angular velocities of the gears about the movable carrier) is determined in the same way as for gears with fixed axes, from the gear ratio

$$i_{31} = \frac{\omega_{3\,\mathrm{rel}}}{\omega_{1\,\mathrm{rel}}} = \frac{z_1}{z_2}\,\frac{z_2}{z_3} = \frac{z_1}{z_3}$$

(gear *2* is an idler and so does not affect the gear ratio), whence

$$\omega_{3\,\mathrm{rel}} = \frac{z_1}{z_3}\,\omega_{1\,\mathrm{rel}}.$$

The absolute angular velocity of the fixed gear *1* is zero. Now the relative angular velocity $\omega_{1\,\mathrm{rel}}$ of this gear may be found from the following considerations. If we fix the observer in the carrier (moving frame of reference), then provided the rotation of the carrier with angular velocity ω_H is in one direction, any radius of gear *1* will appear to that observer as rotating, relative to the carrier, in the opposite sense with the same angular velocity $\omega_{1\,\mathrm{rel}} = \omega_H$. Hence

$$\omega_{3\,\mathrm{rel}} = \omega_{1\,\mathrm{rel}} \frac{z_1}{z_3} = \omega_H \frac{z_1}{z_3}.$$

From Fig. 211 it is readily seen that the relative rotation of gear *3* is in the same direction as its migratory rotation (the rotation of the carrier). Consequently, we have a case of the composition of two rotations about parallel axes in the same direction (Sec. 78). In this case,

$$\omega_3 = \omega_{3\,\text{mig}} + \omega_{3\,\text{rel}} = \omega_H + \omega_H \frac{z_1}{z_3} = \omega_H \left(1 + \frac{z_1}{z_3}\right).$$

Third method (the stop method, or the Willis method). Suggested by the English scientist R. Willis, this method consists in the following. Mentally stop the carrier and impart to the frame of the mechanism and to all its members a rotation with angular velocity equal in magnitude and opposite in direction to the angular velocity ω_H of the carrier. Then the carrier and the axes of all gears mounted on it may be regarded as fixed, and the planetary gear train may be looked upon as an ordinary case of multiple gearing. Then the angular velocities of all gears of the resulting gear train with fixed axes will be equal to the difference between their actual angular velocities and the angular velocity of the carrier.

We thus obtain the following formula (*Willis' formula*) for the gear ratio of a planetary (and differential) spur gear train:

$$\frac{\omega_1 - \omega_H}{\omega_n - \omega_H} = (-1)^k \mid i_{1n} \mid. \qquad (125)$$

In this formula, ω_1 and ω_n are the algebraic values of the angular velocities of the 1st and *n*th gears. Assuming the angular velocity of the carrier to be positive, we will regard the angular velocity of the gear as positive, too, if it rotates in the same direction as the carrier, and negative if it rotates in the opposite direction. ω_H is the angular velocity of the carrier, *k* is the number of externally meshed gears, and i_{1n} is the gear ratio of the gear drive with fixed axes.

Given in this problem are: $\omega_1 = 0$, $k = 1$. Substituting these values into (125), we get

$$\frac{0 - \omega_H}{\omega_3 - \omega_H} = -\frac{z_3}{z_1}.$$

Solving this equation, we get

$$\omega_3 = \omega_H \left(1 + \frac{z_1}{z_3}\right) = 15 \left(1 + \frac{70}{30}\right) = 50 \text{ rad/s}.$$

Since the value of the angular velocity of gear *3* is positive, it must be rotating in the same direction as the carrier.

From the foregoing it is evident that the "stop" method yields the shortest and simplest solution of the problem. This is the method usually used in a kinematic analysis of planetary and differential gears.

Problem 92. Solve the preceding problem on the condition that mechanism shown in Fig. 211 is a differential gear, that is, its central gear *1* is rotated independently from the rotation of the carrier *H* with angular velocity $\omega_1 = 24$ rad/s. Solve the problem for two cases:

(1) gear *1* is in rotation in the same direction as the carrier, and (2) gear *1* is in rotation in the opposite direction to that of the carrier.

Solution. By formula (125) (*Willis' formula*), $\frac{\omega_1-\omega_H}{\omega_3-\omega_H} = (-1)^k \times$ $\times \mid i_{13} \mid$. In our case, $k = 1$, $\mid i_{13} \mid = \frac{z_3}{z_1}$, whence $\frac{\omega_1-\omega_H}{\omega_3-\omega_H} = -\frac{z_3}{z_1}$ and $\omega_3 = \omega_H \left(1 + \frac{z_1}{z_3}\right) - \omega_1 \frac{z_1}{z_3}$.

To find the angular velocity ω_3 of the third gear (of the driven shaft), substitute into the resulting expression $\omega_H = 15$ rad/s, $\frac{z_1}{z_3} = \frac{70}{30} = \frac{7}{3}$, and the algebraic value of the angular velocity of the first gear, $\omega_1 =$ $= 24$ rad/s (when that gear rotates in the same direction as the carrier) or $\omega_1 = -24$ rad/s (when the first gear rotates in the opposite direction to the carrier).

Substituting the given values in the above-found expressions for the angular velocity of the third gear, we find:

(1) If the gear is in rotation in the same direction as the carrier, then

$$\omega_3 = 15\left(1+\frac{7}{3}\right) - 24 \times \frac{7}{3} = -6 \text{ rad/s}.$$

This means that in the given case gear *3* is in rotation in the opposite direction to that of the carrier and with an angular velocity of 6 rad/s.

(2) If the gear rotates in the direction opposite that of the carrier, then

$$\omega_3 = 15\left(1+\frac{7}{3}\right) - (-24) \times \frac{7}{3} = 106 \text{ rad/s}.$$

And so in this case gear *3* rotates in the same direction as the carrier and has an angular velocity of 106 rad/s.

PART III

DYNAMICS

CHAPTER XVII

INTRODUCTION TO DYNAMICS

Sec. 81. The subject of dynamics and its two basic problems

Dynamics is the division of theoretical mechanics that deals with the relationship between the mechanical motion of bodies and the forces acting on them.

In dynamics, the mechanical motion of a body is studied in connection with the physical factors that determine the nature of that motion. Therein dynamics differs from kinematics, where motion is studied solely from the geometric standpoint.

The study of dynamics ordinarily begins with the study of the motion of the simplest object, a particle (material point).

Recall (see Sec. 1) that *a particle (or material point) is a material body whose dimensions can be ignored under the conditions of a given problem.*

Bear in mind that a particle may have infinitesimal dimensions but it may also have finite (and at times very great) dimensions, provided that these dimensions play no part under the conditions of the given investigation. For example, in the case of translational motion, all the points of the body move in the same way, as we know, and to determine the motion of the body it suffices to have a knowledge of the motion of any one of its points, say, the centre of gravity of the body. Later on it will be demonstrated that the centre of mass (this concept will be investigated in Sec. 100) of a body moves like a particle in which the whole mass of the body is concentrated and to which all acting forces are applied. For this reason, *in dynamics a body in translational motion may be regarded as a particle of mass equal to the mass of the body.*

In cases where the dimensions of a body not in translational motion cannot be ignored, we may mentally divide the body into parts that are small compared with the distances that play a role in the given problem, and those parts may be regarded as particles. Hence, any body and any combination of associated bodies may be regarded as a collection of particles.

A mentally conceived collection of interacting particles is termed a mechanical system of particles or, simply, a system.

An *absolutely rigid body may* also *be regarded as* a system of particles the distances between which do not change; what we have then is an *invariable system.*

The multitude of special problems of dynamics may be separated into two basic groups, or reduced to two basic problems.

First problem of dynamics. The motion of a given particle or a given system is known. It is required to determine the forces acting on the particle or system.

Second problem of dynamics (the inverse of the first problem). The forces acting on a given particle or system are known. It is required to find the motion of the particle or system.

In dynamics, these problems are worked by the procedures (established in statics) of the composition of forces and the reduction of their systems to an elementary form and also by the characteristics and means (established in kinematics) of describing various motions. However, all this does not prove sufficient to establish a relationship between the motion of material bodies and the factors that determine such motion; therefore, in dynamics use is made of a number of other physical concepts (mass, momentum, work, energy and so forth). The quantitative relationships between the various physical quantities associated with mechanical motion of material bodies are established in dynamics by mathematical derivation from the basic laws of classical mechanics.

These laws serve as the foundation on which the whole edifice of dynamics rises. We therefore begin the study of dynamics with the basic laws.

Sec. 82. The basic laws of dynamics

First Law (law of inertia)

We examined this law in the introduction to mechanics (Sec. 3). Here we take note of only one circumstance that is of fundamental importance to classical dynamics.

From kinematics we know that the concepts of motion and rest are relative notions and that by referring one and the same motion to different reference frames we can (generally) observe quite distinct motions. For instance, a body at rest on the deck of a sailing vessel or in rectilinear and uniform motion along the deck is in nonrectilinear and nonuniform motion with respect to the riversides as the direction and magnitude of the velocity of the vessel change. In this case, if the law of inertia is applied to the observed motions of the given body, an observer on the deck of the ship and an observer on the riverside will come to completely different conclusions. The former will say the forces applied to the body are balanced, the latter, that equilibrium is absent.

From this example it follows that when applying the laws of classical dynamics, one must first of all decide in what reference frame the laws are applicable (the law of inertia for example).

The reference frame with respect to which the law of inertia holds is the basic (or inertial) frame of reference, and the motion observed with respect to that frame is termed absolute motion.

In kinematics we considered the motions of bodies irrespective of the forces acting on them and for that reason any frame of reference could be taken as fixed and the motion relative to it could be regarded as absolute. In dynamics the situation is different. The subject of dynamics is the relationship between the observed motion of bodies and the forces acting on them; now, this relationship depends on the choice of the reference frame.

When Newton described the basic laws of classical mechanics, he pointed out that they refer to absolute motion, which he perceived to be motion in a certain absolute space. He stated that absolute space in its very essence, irrespective of any external thing, always remains fixed and the same ... and that absolute motion is the displacement of a body from one absolute position to another. It is clear that from the standpoint of dialectical materialism and the present state of science, the concept of an absolutely fixed (dissociated from any matter) "empty" space does not correspond to reality. Since there are no such things as absolutely fixed bodies in nature, so also there is not such thing as a fixed reference frame, with respect to which we could regard motion as absolute in the Newtonian sense of the word.

However, it is possible to find a reference frame that for all practical purposes may be regarded as fixed for all bodies of our solar system.

If we place the coordinate origin at the centre of the sun, or, more precisely, at the centre of mass of the solar system, and send the coordinate axes to some three "fixed" (fixed, in contrast to the perceivably moving planets of our solar system) stars, then we obtain a system of coordinates that goes by the name *heliocentric*.

Experience and observations have shown that for motions (with velocities appreciably less than the velocity of light) referred to the heliocentric system of coordinates, the law of inertia holds true to a very high degree of accuracy and therefore such a reference frame may be regarded as an inertial frame.

If a particle is moving by inertia (that is, rectilinearly and uniformly) relative to an inertial frame with an absolute velocity $\mathbf{v}$, then such will be the motion of that particle relative to any other reference frame moving with respect to the inertial frame translationally, rectilinearly and uniformly with velocity $\mathbf{v}_{\mathrm{mig}}$.

Indeed, as follows from the theorem on the composition of velocities, the relative velocity of a particle $\mathbf{v}_{\mathrm{rel}} = \mathbf{v} - \mathbf{v}_{\mathrm{mig}}$.

But in this instance, $\mathbf{v}$ and $\mathbf{v}_{\mathrm{mig}}$ are constant vectors, and so $\mathbf{v}_{\mathrm{rel}}$ is also a constant vector.

From this it follows that *any frame of reference that exhibits. translational, rectilinear and uniform motion relative to an inertial frame will also be an inertial frame.* Now any frame of reference that is in nonrectilinear motion or even in rectilinear motion but nonuniformly with respect to an inertial frame is no longer an inertial frame.

Since the earth moves about the sun in a certain curvilinear orbit, at the same time rotating on its axis, then strictly speaking, a frame of reference attached to the earth is not an inertial frame. However, due to the small curvature of the terrestrial orbit* and the small angular velocity of rotation of the earth on its axis (one rotation every 24 hours), *in most problems of dynamics* encountered in ordinary engineering practice we *may, to an entirely sufficient degree of accuracy, regard the inertial frame of reference as fixed with respect to the earth.* Corrections are required only in those relatively rare instances where the earth's rotation cannot be disregarded: in problems of artillery fire and long-range rockets, in the study of marine and air currents and certain other high-speed and long-term motions.

Second Law (basic law of dynamics)

The acceleration imparted to a particle by an applied force is proportional to the magnitude of the force and coincides with it in direction.

One has to bear in mind that this statement assumes the particle to which the given force is applied to be completely free, that is, it does not encounter any obstacles to its motion.**

It must be stressed that it is the *direction of acceleration* (*and not the direction of motion* or the direction of velocity) *that always coincides with the direction of the force.* The direction of motion may not coincide with the direction of the applied force. To illustrate, take a particle thrown in a vacuum at an angle to the horizon. It moves in a curve (parabola), all the time changing the direction of its motion, whereas the force of gravity acting on the particle (and imparting to it an acceleration) is always directed vertically downwards.

---

* In its orbital motion about the sun, the earth describes in one hour an arc only slightly greater than 2′.

** As we shall see later on, a force applied to a particle is to be understood as the resultant of all forces acting on the particle (and these include the forces of reaction constraints if the particle is not free).

The direction of motion will coincide with the direction of the force only if the force acts on a free particle originally at a state of rest. Now, if a particle is already in motion, the acceleration imparted to it by a force only alters the velocity, and the direction of the velocity may not coincide with the direction of the force.

From everyday experience we know that one and the same force imparts different accelerations to different bodies (even if the bodies have the same shape and dimensions but are of different material). Thus, the magnitudes of accelerations experienced by different bodies depend not only on the magnitudes of the acting forces but also on a certain property of the bodies themselves. This property is characterized by a special physical quantity called mass.

Newton defined the *mass of a body as the quantity of matter contained in the body.*

The weight of a homogeneous body is proportional to the volume of the body and hence to the quantity of matter in it. The method of comparing the masses of bodies by their weight is extended to inhomogeneous bodies: the masses of any two bodies are said to be the same if, when weighed at the same place, they have the same weight. It will be recalled that the weight of a body varies with the geographic latitude at which the body is weighed. Now, the mass of a body, according to classical mechanics, is an invariable quantity and therefore the weight of a body cannot be taken as an exact measure of its mass.

It is also known that at every point of observation in a vacuum, all bodies fall with the same acceleration. The magnitude of that acceleration, g, depends on the locality in the same way that the weight does, so that the ratio of the weight of a body to the acceleration of a freely falling body is a constant quantity. That relation is taken as the measure of the mass of the body.

Denoting the mass of a body by m, its weight by G and the acceleration of a freely falling body by g, we have

$$m = \frac{G}{g}. \tag{126}$$

The mass of a body is equal to the ratio of its weight to the acceleration of a freely falling body at the site where the body was weighed.

Since the acceleration of a freely falling body does not depend on its dimensions, the mass of a particle is given by its weight via the same relation (126) as the mass of any body.

Suppose a free particle of weight G is acted upon by a force P that imparts an acceleration a. According to the law under consideration, the magnitudes of the accelerations imparted to the particle by applied forces must be proportional to the magnitudes of the forces.

Thus,

$$\frac{P}{G} = \frac{a}{g},$$

whence we have

$$P = \frac{G}{g} a$$

or, by (126),

$$P = ma. \tag{127}$$

The magnitude of a force applied to a particle is equal to the product of the mass of the point by the magnitude of its acceleration.

Equality (127) yields a dynamic method for determining the magnitude of a force.

From this equation it follows that $a = P/m$, which means that the greater the mass of the given particle, the smaller the acceleration of the particle imparted by the given force. Hence, the greater the mass of a particle, the slower the change in the velocity of the particle (under the action of an applied force), and the smaller the deviation of its motion from the inertial motion. Thus, different particles have different inertia and the *measure of the inertia of a particle is its mass.*

Since different points of a rigid body may have different motions and different accelerations, the mass of a body is not always a measure of its inertia. Generally, the inertia depends not only on the value of the masses of the particles of the body but also on their distribution in the body. *The mass of a body fully characterizes its inertia only when the body is in translational motion* (that is, when accelerations of all particles of the body are the same).

Since the direction of a force always coincides with the direction of the acceleration imparted by the force to a free particle, and the mass of the particle is a positive scalar quantity, it follows that (127) may be cast in the form of a vector equation:*

$$\mathbf{P} = m\mathbf{a}. \tag{128}$$

The vector of a force applied to a particle is equal to the product of the mass of the particle by the vector of its acceleration.

Equation (128), which establishes the relationship between the motion of a particle and the force acting on it and which is a complete mathematical statement of the basic law of dynamics, *is termed the basic equation of dynamics of a particle.*

With a change in the reference frame, the observed nature of motion of the particle and, hence, its acceleration may vary; for this reason, the second law of dynamics, like the first law, cannot be applied irrespective of the reference frame.

By the acceleration of a particle that enters into the basic equation of dynamics one must understand the absolute acceleration of the particle, that is, its acceleration relative to the frame of reference assumed to be the inertial frame.

Third Law (the law of equality of action and reaction)

The forces with which two particles act on each other are always equal in magnitude and opposite in direction (along a single straight line joining the given particles).

---

* Multiplying a vector by a positive scalar m yields a vector of the same direction but differing in magnitude by a factor of m.

This law was examined in detail in the introduction to mechanics (Sec. 3) and was widely used in statics.

If a particle A acts on a particle B with a force $\mathbf{P}_A$, then particle B acts on particle A with a force $\mathbf{P}_B = -\mathbf{P}_A$ (Fig. 212). Suppose the mass of particle A is m_1 and the acceleration imparted by the force $\mathbf{P}_B$ is $\mathbf{a}_1$, while the mass of particle B is m_2 and the acceleration imparted to it by the force $\mathbf{P}_A$ is $\mathbf{a}_2$. By the basic equation of dynamics, $P_B = m_1a_1$ and $P_A = m_2a_2$. But according to the given law, $P_A = P_B$ and so $m_1a_1 = m_2a_2$, whence we have

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}. \qquad (129)$$

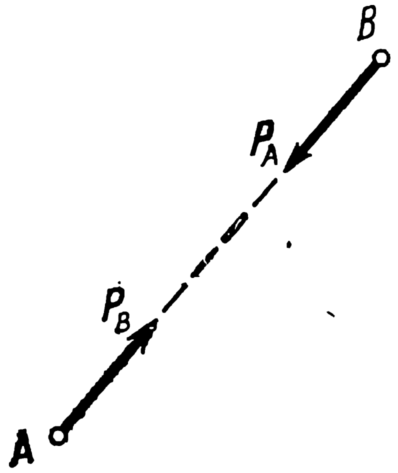


FIG. 212

The magnitudes of the accelerations imparted to each other by two particles are inversely proportional to the masses of the particles. And, like the forces of interaction, these accelerations are directed along a single straight line AB in opposite directions.

Fourth Law (the law of the independence of action of forces)

The acceleration acquired by a particle acted upon simultaneously by several forces is equal to the geometric sum of the accelerations that the particle would have received under the action of each separate force.

Suppose a particle of mass m is simultaneously acted upon by forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, . . ., $\mathbf{P}_n$ which impart an acceleration $\mathbf{a}$ to the particle. The accelerations imparted by each of the given forces separately would be $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, . . ., $\mathbf{a}_n$. According to the given law, which was established by Galileo on the basis of numerous experiments, we have

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \ldots + \mathbf{a}_n. \qquad (130)$$

If we multiply both sides of this equation by the scalar factor m (the mass of the particle), we obtain

$$m\mathbf{a} = m\mathbf{a}_1 + m\mathbf{a}_2 + m\mathbf{a}_3 + \ldots + m\mathbf{a}_n.$$

By the basic equation of dynamics (128),

$$m\mathbf{a}_1 = \mathbf{P}_1, \quad m\mathbf{a}_2 = \mathbf{P}_2, \quad m\mathbf{a}_3 = \mathbf{P}_3, \ \ldots, \ m\mathbf{a}_n = \mathbf{P}_n,$$

whence we get

$$m\mathbf{a} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \ldots + \mathbf{P}_n.$$

Denoting by $\mathbf{P}$ the resultant of the system of forces applied to the given particle, which is equal to their geometric sum, we have

$$m\mathbf{a} = \mathbf{P}.$$

Outwardly, this equation does not differ at all from the basic equation of dynamics (128). This means that the *basic equation of dynamics also holds true when a particle is simultaneously acted upon by several forces. The force* **P** *applied to the particle must be regarded here as the resultant of all forces acting on the particle.*

Sec. 83. Systems of units

The units needed to measure all physical quantities that occur in theoretical mechanics may be expressed in terms of three basic units which taken together determine the system of units. In all systems, two basic units are the unit of length and the unit of time. The third basic unit is either the unit of force or the unit of mass. The relationship between the magnitude of force and the mass is expressed by the basic equation of dynamics:

$$P = ma. \qquad (131)$$

As we know, acceleration has the dimensions of $\frac{[\text{length}]}{[\text{time}]^2}$, which means that the unit of acceleration is a derived unit* of two basic units: length and time.

Hence, to measure the other two quantities (force and mass) that appear in the relation (131) we can take for the basic (independent) unit only one of them; the unit of measurement of the other will then be a derived unit of the three basic units.

---

* Derived units are units that are "derived" from basic units by means of mathematical operations that express certain physical laws.

The use of different systems of units not only in different countries but even in different areas of science and engineering creates appreciable difficulties and therefore the 11th General Conference on measures and weights in 1960 approved a new "International System of Units" (abbreviated SI which in French stands for Système International d'Unités).

At the present time, the USSR and several other countries are introducing this system as an obligatory system and a national standard. In the SI, a basic unit is the unit of mass (kg) equal to the mass of the international prototype of the kilogram.* The dimensions of force in this system are obtained from the basic equation of dynamics:

$$[\text{force}] = [\text{mass} \times \text{acceleration}] = \left[\frac{\text{mass} \times \text{length}}{\text{time}^2}\right]$$

In the SI, the unit of force is the newton (N)**. To express this unit in terms of the basic units, substitute into the formula of its dimensions: the unit of mass (1 kg—kilogram), the unit of length (1 m—metre) and the unit of time (1 s—second).

We thus have

$$1\ \text{N} = \text{kg}\cdot\text{m/s}^2.$$

The newton is a force that imparts to a mass of 1 kg *an acceleration of* 1 m/s². Still in use is the so-called *system of units in which the basic unit is force*: kilogram force (kgf) equal to the weight of the International Prototype Kilogram.

The dimension of mass in the engineering system is found from the basic equation of dynamics:

$$[\text{mass}] = \left[\frac{\text{force}}{\text{acceleration}}\right] = \left[\frac{\text{force} \times \text{time}^2}{\text{length}}\right]$$

In the engineering system of units, the unit of mass is the technical unit of mass (t.u.m.):

$$1\ \text{t.u.m.} = \text{kgf}\cdot\text{s}^2/\text{m}. \qquad (132)$$

---

* The International Prototype Kilogram is a platinum-iridium standard kept at the International Bureau of Measures and Weights. Its mass is approximately 0.003% greater than the mass of 1 cubic decimeter of distilled water at 4 °C.

** In honour of one of the founders of classical mechanics, Isaac Newton (1643-1727).

The variation of the acceleration g of a freely falling body is slight for different points of the earth's surface and ordinarily the mean value is used: $g = 9.81$ m/s^2.

If the weight of a body is $G = 9.81$ kgf, its mass is

$$m = \frac{G}{g} = \frac{9.81}{9.81} = 1 \text{ kgf} \cdot \text{s}^2\text{m}. \tag{133}$$

Hence, *the technical unit of mass is equal to the mass of a body whose weight is equal to* 9.81 kgf.

In problem solving we will use the SI. But since the engineering system will obviously be used for some time to come and we will be encountering the units of that system, we will express derived units that come up in various parts of dynamics both in the SI and the engineering system. We will also give the relationship that exists between them. The derived units are determined very simply if we know the relationship between the unit of force in the SI (it is N) and the unit of force in the engineering system (kgf).

A force of 1 kgf is the weight of the standard having a mass of 1 kg. From the formula $G = mg$ it follows that 1 kgf = 1 kg × 9.81 m/s^2 = 9.81 kg·m/s^2 = 9.81 N. And so we have a force of 1 kgf = 9.81 N, or a force of 1 N = = 0.102 kgf.

Problem 93. A body of mass $m = 10$ kg is in rectilinear motion over a smooth horizontal plane under the action of a horizontal force **P**. The equation of motion of the body is of the form $s = 4t + 2t^2$ (s in metres, t in seconds). Find the magnitude of the force **P** in newtons and in kilograms.

Solution. If the body is in rectilinear translational motion, its acceleration is

$$a = \frac{d^2s}{dt^2} = (4t + 2t^2)'' = 4 \text{ m/s}^2.$$

By the basic equation of motion of a particle,

$$P = ma = 10 \times 4 = 40 \text{ kg} \cdot \text{m/s}^2 = 40 \text{ N}$$

or

$$P = 40 \times 0.102 \text{ kgf} = 4.08 \text{ kgf}.$$

Problem 94. Referring to Fig. 213, a body weighing G is descending (under the action of gravity) an inclined plane that forms an angle of $\alpha = 30^\circ$ with the horizon. Find the velocity of the body 2 seconds after the start of motion and determine the path covered in that time if the coefficient of friction $f = 0.3$.

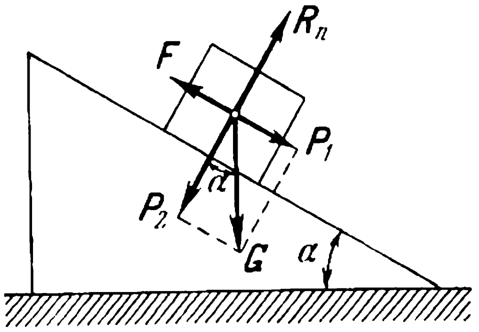


FIG. 213

Solution. Since the body is in translational motion, it may be regarded as a material point (particle) with the mass concentrated at the centre of gravity of the body and with all forces acting on the body being applied to that centre. The forces acting on the body are: the force of gravity **G** of the body, the normal reaction of the plane $\mathbf{R}_n$ and the force of friction **F**. We resolve **G** into two components: a force $\mathbf{P}_1$ parallel to the plane and a force $\mathbf{P}_2$ perpendicular to the plane and balanced by the normal reaction $\mathbf{R}_n$ of the plane. The force of sliding friction of the body over the plane is $F = fP_2$, where the normal pressure of the body on the plane is $R_n = P_2 = G \cos \alpha$. Consequently, $F = fG \cos \alpha$. Thus, the resultant of all forces applied to the body is

$$P = P_1 - F = G \sin \alpha - fG \cos \alpha$$
$$= G (\sin \alpha - f \cos \alpha) = mg (\sin \alpha - f \cos \alpha).$$

By the basic equation of dynamics, the acceleration of the body is

$$a = \frac{P}{m} = \frac{mg (\sin \alpha - f \cos \alpha)}{m} = g (\sin \alpha - f \cos \alpha)$$
$$= 9.81 (0.5 - 0.3 \times 0.866) \approx 2.4 \text{ m/s}^2$$

Since the acceleration is a constant, the motion of the body is uniformly accelerated motion. By the formulas of kinematics we find

$$v = v_0 + at = 0 + 2.4 \times 2 = 4.8 \text{ m/s},$$

$$s = v_0 t + \frac{at^2}{2} = 0 + \frac{2.4 \times 2^2}{2} = 4.8 \text{ m}.$$

CHAPTER XVIII

ESSENTIALS OF KINETOSTATICS (DYNAMIC FORCE ANALYSIS)

Sec. 84. D'Alembert's principle

The principle associated with the name of the outstanding French scientist Jean d'Alembert (1717-1783) underlies an important method of dynamics that permits problems of dynamics to be reduced formally to problems of statics.

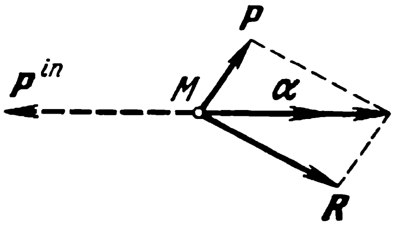


FIG. 214

Suppose a constrained* material point (particle) M of mass m is acted upon by an active force **P**. Mentally removing the point from the constraints, we replace their action on the particle by the reaction force **R** of the constraints. Then the particle M may be regarded as being free but under the action of the force **P'**, which is the resultant of the forces **P** and **R** (Fig. 214). By the basic equation of dynamics of a particle, $\mathbf{P}' = m\mathbf{a}$. Let us agree also to apply to M the vector

$$\mathbf{P}^{\text{in}} = -m\mathbf{a}. \qquad (134)$$

The vector $\mathbf{P}^{\text{in}}$, *which is equal to the product of the mass of the particle by its acceleration and is in the opposite direction to that of the acceleration of the particle**, is termed the force of inertia of the particle.* Referring to Fig. 214, the geometric sum of the forces **P'** and $\mathbf{P}^{\text{in}}$ or, what is the same thing, the geometric sum of the forces **P**, **R** and $\mathbf{P}^{\text{in}}$ is equal to zero, obviously, and this collection of forces is consequently a balanced system.

---

* A constrained particle is one whose motion is restricted by imposed constraints.

** Note that the vector $\mathbf{P}^{\text{in}}$ of the force of inertia of a particle is always in the opposite direction to the vector of acceleration of the particle, and not to the direction of its motion. In the case of rectilinear uniformly decelerated motion of the particle, the direction of the vector $\mathbf{P}^{\text{in}}$ coincides with the direction of motion of the particle.

The force of inertia $\mathbf{P}^{in}$ *of a particle is conditionally applied to the particle itself, and therefore the resulting equilibrium is not real but imaginary.*

However, this conditional adjoining of the force of inertia of a particle to the number of forces actually impressed permits applying to dynamical problems familiar procedures of statics and it underlies the so-called method of dynamic force analysis.

The idea behind the method of dynamic force analysis may be formulated* for a particle in the following manner: *at any instant of time in the motion of a particle, the active forces applied to the particle, the reaction forces of the constraints imposed on it, and the force of inertia of the given particle (conditionally applied to the particle itself) mutually balance.*

This method may readily be extended to a material system. Imagine a moving system made up of particles associated by means of some law. At any time of motion of the system, the active forces applied at some point, the forces of reaction of the constraints, and the force of inertia of the point (which force is conditionally applied to the point itself) will mutually balance. From this it follows that the whole collection of forces will be balanced, the active forces applied to the given system, the forces of constraint reactions, and the forces of inertia of all particles of the system.

Thus, *at any time of motion of the system, the active forces applied to it, the forces of constraint reactions, and the conditionally applied (to every point of the system) forces of inertia of the particles mutually balance.*

To summarize then: any particle and any system of particles may, when employing the method of dynamic force analysis, be regarded at any time of their motion to be in equilibrium (imaginary, of course) and, hence, for every specific case of the arrangement of force, one can set up the appropriate number of independent equations of equilibrium, just as is done in statics. Because of its simplicity

---

** This statement is a widely used modern formulation of the d'Alembert principle, although the forces of inertia did not occur in the statement of the principle given by d'Alembert.

and clarity, the method of dynamic force analysis is widely used in engineering practice for solving problems of dynamics.' This method is particularly convenient for determining the so-called *dynamic reactions of constraints, that is, reactions that appear in constraints during the motion of a system.* This method can also be employed to find the accelerations of bodies making up a system.

It is also necessary to point out that all problems of dynamics (without exception) may be worked without resorting to the method of dynamic force analysis, thus dispensing completely with the concept of forces of inertia.

Sec. 85. The concept of an inertial force

There is no such thing as one-sided action in nature. If a particle with mass m acquires a certain acceleration $\mathbf{a}$ due to interaction with other surrounding bodies, then by the third basic law of mechanics, to these bodies are applied, by the particle, forces of reaction. The geometric sum of these forces which, generally speaking, are applied to different bodies, is formally equal to $-m\mathbf{a}$, that is, it is equal to the force of inertia of the particle. Only formally, since any composition of forces applied to distinct bodies is physically meaningless.

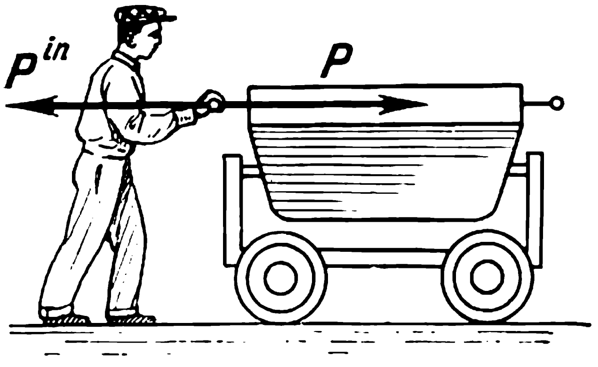


FIG. 215

It is only the components of this force that have real existence. Components applied to those bodies which are the source of the active forces acting on the particle undergoing acceleration, and components applied to the constraints imposed on the particle. We can speak of the force of inertia, $\mathbf{P}^{\text{in}} = -m\mathbf{a}$, as a real force provided that only one body acts on the particle.

Let us consider an illustration. Referring to Fig. 215, a worker is pushing a car along rails, thus imparting to it an acceleration $\mathbf{a}$. For the sake of simplicity, we will assume that the car is moving on a horizontal rectilinear track and is not experiencing any resistance whatsoever. Thus, the sole force imparting to the car an acceleration $\mathbf{a}$ comes from the worker. By the basic law of dynamics, in order to impart that acceleration, the worker has to apply to the car a force $\mathbf{P} = m\mathbf{a}$, where m is the mass of the car. Every action of one body on another is always accompanied by a reaction (in the opposite direction) of the latter body on the former. Consequently, the worker will experience a reaction force (from the car) $\mathbf{P}^{\text{in}} = -m\mathbf{a}$, which is equal in magnitude to the force $\mathbf{P}$ of pressure of the worker on the car and is in the opposite direction (opposite to the acceleration). It is obvious that the force $\mathbf{P}^{\text{in}}$ of the car is applied to the hand of the worker and not to the car itself.

It is necessary to bear in mind when employing the method of dynamic force analysis that we only imagine that the force of inertia of the particle is applied to the particle itself. If equal and opposite forces $\mathbf{P}$ and $\mathbf{P}^{\text{in}}$ were indeed applied to one particle, they would always balance, and the particle would be at rest or would be moving uniformly in a straight line, that is, its acceleration would always be zero. Obviously, in that case the force of inertia itself, $\mathbf{P}^{\text{in}} = -m\mathbf{a}$, would always be zero as well.

We know that when a particle maintains its velocity in the absence of other bodies acting on it or if such bodies are in equilibrium, this property is termed the *inertia of the particle.* From this, the force $\mathbf{P}^{\text{in}} = -m\mathbf{a}$ that arises when the motion of the particle changes is called an inertia (force of inertia), the name being well established but not particularly apt since it leads to a certain confusion of the concepts inertia and force of inertia.

A particle always possesses the property of inertia, whereas the force of inertia appears only when the particle is acted upon by other bodies that make the particle change its inertial motion. And, as we have already mentioned above, in the process there is no force of inertia of the particle actually applied to the particle.

Sec. 86. Forces of inertia in the curvilinear motion of a particle

In the general case of the motion of a particle in a curvilinear trajectory, the acceleration $\mathbf{a}$ of the particle is, as we know, conveniently decomposed into two components: the tangential acceleration $\mathbf{a}_t$ directed along the tangent to the trajectory of motion, and the normal accelaration a_n, which is directed along the normal towards the concavity of the trajectory, that is, towards the centre of curvature. Referring to Fig. 216, let us suppose that a force P is applied to a free particle M of mass m moving with velocity v, the direction of the force forms a certain angle with the direction of the velocity v. In this case, the particle will move along a curvilinear trajectory with acceleration $\mathbf{a} = \mathbf{P}/m$ and in the same direction as the force **P**. Let us resolve it into component accelerations: the tangential acceleration which is numerically equal to $a_t = \frac{dv}{dt}$—the derivative of the magnitude of the velocity of the particle with respect to time, and the normal acceleration, which is equal in magnitude to $a_n = \frac{v^2}{\rho}$, where ρ is the radius of curvature of the trajectory for the given position of the particle.

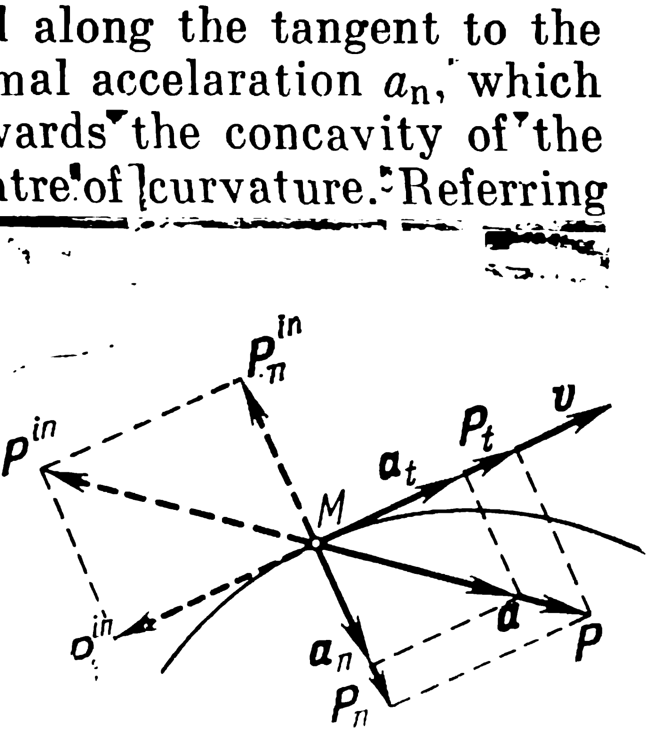


FIG. 216

It is obvious that *in curvilinear motion of the particle the force* **P** *applied to it may be resolved into two components: a tangential force* $\mathbf{P}_t = m\mathbf{a}_t$ *that change the magnitude of the velocity of the particle, and a normal (centripetal) force* $\mathbf{P}_n = m\mathbf{a}_n$, *which changes the direction of the velocity of the particle.* This latter component compels the particle to deviate from a straight line in the direction of the tangent to the path at a given point towards the respective centre of curvature, whence the name centripetal force.

In the curvilinear motion of a particle, the force of inertia of the particle may also be resolved into two components: a tangential force of inertia $\mathbf{P}_t^{in} = -m\mathbf{a}_t$ *directed opposite to the tangential acceleration of the particle, and a normal (centrifugal) force of inertia* $\mathbf{P}_n^{in} = -m\mathbf{a}_n$ *directed opposite to the normal acceleration of the particle.*

Since a normal acceleration is directed along the normal to the trajectory towards its convexity, the centrifugal force of inertia (or, simply, the centrifugal force) is directed along the normal towards the convexity of the trajectory, that is, along the normal away from the centre of curvature. Whence the name of this force. Clearly, *the centrifugal force is equal in magnitude and opposite in direction to the centripetal force.*

If a particle M is in uniform motion along a curvilinear trajectory, then $v =$ constant and $a_{\mathrm{t}} = dv/dt = 0$. In this case, the tangential force of inertia vanishes and the total force of inertia consists solely of the centrifugal component that is equal in magnitude to

$$P^{\mathrm{in}} = P_{\mathrm{n}}^{\mathrm{in}} = \frac{mv^2}{\rho}. \qquad (135)$$

If particle M belongs to a body in rotation about a fixed axis, then, as we know from kinematics, its tangential and centripetal accelerations may be computed from the formulas $a_{\mathrm{t}} = r\varepsilon$ and $a_{\mathrm{n}} = r\omega^2$, where ω and ε are the angular velocity and angular acceleration of rotation of the body respectively, and r is the distance of the particle from the axis of rotation. From this, the magnitudes of the tangential and centrifugal forces of inertia may be computed via the formulas

$$\left.\begin{aligned} P_{\mathrm{t}}^{\mathrm{in}} &= mr\varepsilon, \\ P_{\mathrm{n}}^{\mathrm{in}} &= mr\omega^2. \end{aligned}\right\} \qquad (136)$$

If the body is rotating uniformly, then $\varepsilon = 0$ and $P_{\mathrm{t}}^{\mathrm{in}} = 0$. In this case, the total force of inertia is numerically equal to

$$P^{\mathrm{in}} = P_{\mathrm{n}}^{\mathrm{in}} = mr\omega^2. \qquad (137)$$

In the case of a constrained curvilinear motion of a particle, the centripetal force acting on it is a constraint reaction that forces the particle to deviate from its rectilinear path and, consequently, imparts to it an appropriate normal acceleration. Now the force acting on the constraint is the centrifugal force of inertia of the given particle.

Referring to Fig. 217, suppose a ball is in a curved pipe. If the ball is given a push in the direction of the axis of the pipe, there will appear two forces at once, a centripetal force $\mathbf{P}_{\mathrm{n}}$, the pressure of the walls of the pipe on the ball directed along the normal to the centre of curvature, and a centrifugal force $\mathbf{P}_n^{in}$, the pressure of the ball on the walls of the pipe directed along the same normal but outwards from the centre. If we take a rubber hose, the action of the centrifugal force is clearly evident in that the outer surface of the hose bulges as the ball moves along.

A body attached to a string and suspended (Fig. 218) from a beam stretches the string (at rest) with a force equal in

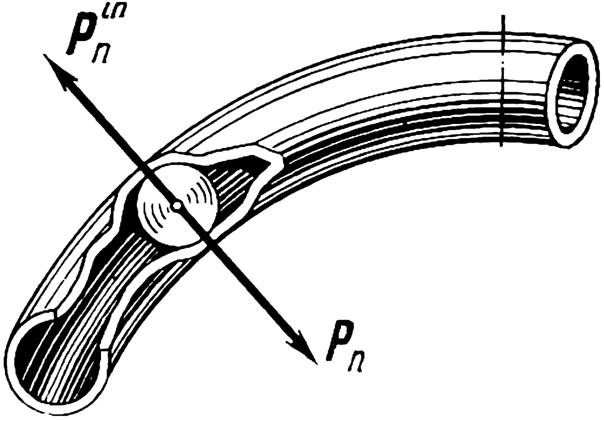


FIG. 217

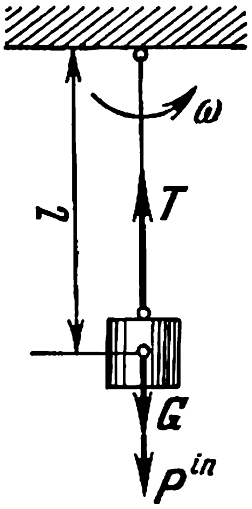


FIG. 218

magnitude to the weight G of the body. Now when the body is set into oscillation, the body stretches the string at the time it passes through the vertical position with a force $\mathbf{T}$, which is equal in magnitude to

$$T = G + P^{\mathrm{in}} = G + \frac{G}{g} r\omega^2.$$

In the case of rapid rotation, the centrifugal force of inertia of the body (this force is applied to the string which is the constraint compelling the body to perform curvilinear motion) may increase the tension of the string to such an extent that the string will break. When the string breaks, the centripetal force applied to the body vanishes because the constraint vanishes that made the motion of the body constrained motion. At the very same instant, the centrifugal force will vanish too, and the body will move along the tangent to the circle at the point at which it was located when the string broke.

A similar action of centrifugal force explains why flywheels break occasionally when rotated at very high speeds. If the whole mass of a rotating body is distributed symmetrically about its axis of rotation, then the centrifugal forces developed by its separate parts result only in the generation of dynamic (internal) stresses in the material of the body. At high speeds, these dynamic stresses may reach very high values and so must definitely be taken into account. Now if the mass of a rotating body is distributed nonsymmetrically about the axis of rotation, then the centrifugal forces of separate particles of the body exert an added pressure on the bearings, thus increasing friction and wear in the bearings. Due to the rotation of the body, the resultant of unbalanced forces of inertia will all the time change its direction, and this gives rise to undesirable vibrations of the body. The problem of balancing the forces of inertia is of great importance in machine-building and is discussed in the theory of mechanisms and machines. When designing a new machine, one has to take into account the forces of inertia that may arise in it under various operating conditions.

Problem 95. Referring to Fig. 219, a load of weight $G=100N$ is being hoisted vertically upwards with an acceleration of $a = 5$ m/s^2 with the aid of a rope passed over a fixed block. Determine the tension of the rope.

Solution. The load is acted on by the following forces: the force of gravity **G** of the load directed vertically downwards, and the reaction **T** of the rope, which is numerically equal to the tension and is directed vertically upwards. Let us now adjoin to these forces the force of inertia $\mathbf{P}^{\text{in}}$ of the load, which is equal in magnitude to ma and is opposite to the direction of acceleration, that is, it is directed vertically downwards. Now, using the method of dynamic force analysis, we may regard the load as being in equilibrium under the action of the forces **G**, **T** and $\mathbf{P}^{\text{in}}$. From the condition of the equilibrium of forces acting along a single straight line, we have

$$G+P^{\text{in}}=T,$$

whence

$$T=G+P^{\text{in}}=G+\frac{G}{g}a=100+\frac{100}{9.81}\times 5\approx 151 \text{ N}.$$

So as to be able to solve the problem by the method of dynamic force analysis, we applied the force $\mathbf{P}^{\text{in}}$ of the load to the load itself. But this force is real with respect to the rope that forces the load to move upwards with a given acceleration. At a certain acceleration, the tension T of the rope may prove to be so great that the rope will break.

If the load dropped with the same acceleration, then its force of inertia would be directed vertically upwards. For the same data, we would have

$$G-P^{\text{in}}=T,$$

whence $T = G - P^{\text{in}} = 100 - \frac{100}{9.81} \times 5 \approx 49$ N.

If the load were merely suspended on the rope (if it were at rest) or if it were moving uniformly in a straight line, then $a=0$ and $T=G=$ $=100$ N.

Problem 96. Referring to Fig. 220, a flexible rope is passed over a pulley and, suspended at the ends of the rope, are loads weighing G_1

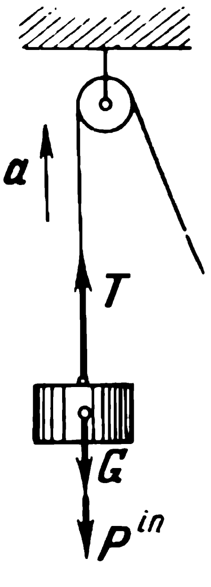


FIG. 219

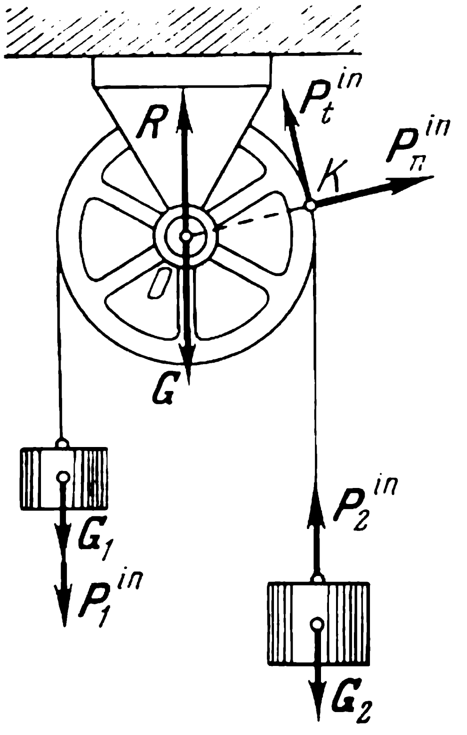


FIG. 220

and G_2, $G_2 > G_1$. Find the acceleration a of the loads and take into account the mass of the pulley. The pulley weighs G and is of radius r. For the sake of simplicity, regard the mass of the pulley as being distributed uniformly along its rim.

Solution. The force of inertia of the greater load $P_2^{\text{in}} = G_2a/g$ and is directed vertically upwards. The force of inertia of the smaller load $P_1^{\text{in}} = G_1a/g$ and is directed vertically downwards. Since the pulley is in rotation, the force of inertia of each of the particles of the pulley (say, particle K) may be decomposed into two components: a tangential force of inertia $\mathbf{P}_t^{\text{in}} = -m\mathbf{a}_t$, which is in the opposite direction to the tangential acceleration, and a centrifugal force of inertia $\mathbf{P}_n^{\text{in}} = -m\mathbf{a}_n$ directed along a radius from the axis of the pulley. Adjoining to the translationally moving loads (regarded as particles) and to all particles of the pulley their forces of inertia, we may regard the whole system as being in equilibrium.

As we know from statics, for the equilibrium of a body having a fixed axis, it is necessary and sufficient that the sum of the moments about the axis of all forces applied to the body be equal to zero.

For a variable system (which is what we have here), this condition will also be necessary but not sufficient.

Setting up the sum of the moments of the forces about the axis of the pulley O, we get

$$G_1r+P_1^{\text{in}}r+P_2^{\text{in}}r-G_2r+\sum P_t^{\text{in}}r=0.$$

The centrifugal forces of inertia $\mathbf{P}_n^{\text{in}}$ and also the force of gravity of the block, $\mathbf{G}$, and the reaction $\mathbf{R}$ of the axis of the pulley do not enter into this equation because their moments about the axis O of the pulley are zero.

We have assumed the mass of the pulley to be uniformly distributed about the rim of the pulley. In the absence of slipping of the rope over the rim, the magnitude a_t of the tangential acceleration of a point on the rim is equal to the magnitude a of the acceleration of the rope (loads) and, hence, the magnitude of the tangential force of inertia of a particle of the rim $P_t^{\text{in}} = ma_t = ma$. The sum of the moments of tangential forces of inertia of all points of the pulley (located on the rim) is

$$\sum P_t^{\text{in}}r=\sum mar=ar\sum m=arM,$$

where M is the mass of the whole rim. But $M = G/g$. Consequently,

$$\sum P_t^{\text{in}}r=\frac{G}{g}\,ar.$$

Putting the values of P_1^{in}, P_2^{in} and $\sum P_t^{\text{in}}\,r$ into the above equilibrium equation of the system, we obtain

$$G_1r+\frac{G_1}{g}\,ar+\frac{G_2}{g}\,ar-G_2r+\frac{G}{g}\,ar=0,$$

whence we find

$$a=\frac{(G_2-G_1)\,g}{G_1+G_2+G}.$$

Problem 97. A cross-beam COD (Fig. 221), at the ends of which are point loads C and D of equal weight ($G_1 = G_2 = G$), is in uniform rotation about a vertical axis AB that passes through the midpoint of the length of the cross-beam. Ignore the mass of the cross-beam and determine the reactions of a thrust bearing A and a cylindrical bearing B when the cross-beam COD lies in the plane xAy. Given: the distances $AO = m$, $BO = n$, $CO = OD = l$ and the fixed angle α between the cross-beam and the axis of rotation AB.

Solution. Using the method of dynamic force analysis, we adjoin to the forces acting on the cross-beam the forces of inertia of the point loads C and D. Since the cross-beam is in uniform rotation ($\varepsilon = 0$), the tangential components of the forces of inertia of the loads are equal to zero. The centrifugal forces of inertia of the loads are equal, in magnitude, to

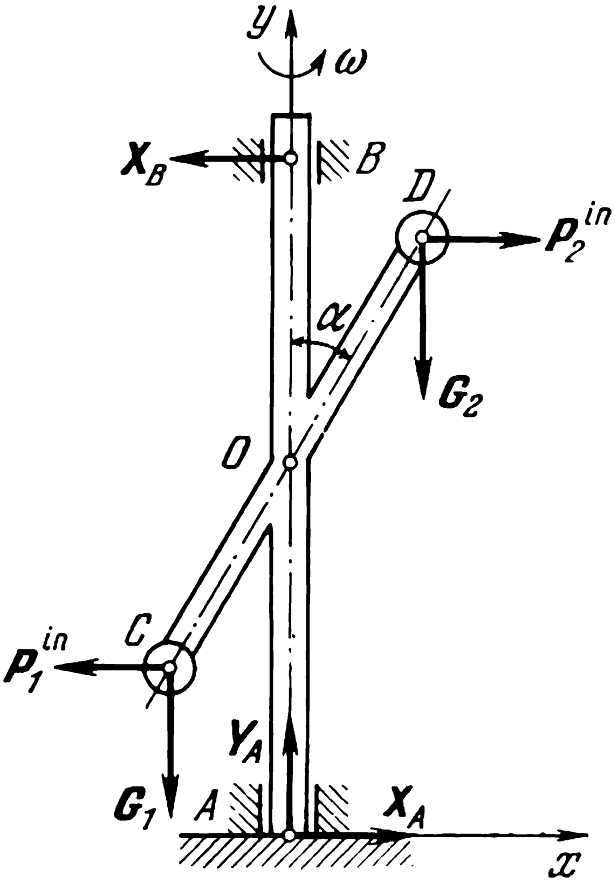


FIG. 221

$$P_1^{\text{in}} = P_2^{\text{in}} = P^{\text{in}} = \frac{G}{g} r\omega^2 = \frac{G}{g} l\omega^2 \sin\alpha.$$

When the cross-beam is in the plane xAy, the active force $\mathbf{G}_1$ and $\mathbf{G}_2$ acting on it and the centrifugal forces of inertia of the loads lie in the same plane. Hence, the constraint reactions will also lie in that plane. The reaction of the thrust bearing (whose direction is unknown) can be decomposed into components: $\mathbf{X}_A$ and $\mathbf{Y}_A$. The reaction X_B of the bearing is perpendicular to its axis.

Let us now set up the equations of equilibrium (conditional, of course) for the plane system of forces

$$\sum X_k = X_A - X_B + P_2^{\text{in}} - P_1^{\text{in}} = 0, \quad \text{whence} \quad X_A = X_B$$

(since $P_1^{\text{in}} = P_2^{\text{in}}$),

$$\sum Y_k = Y_A - G_1 - G_2 = 0, \quad \text{whence} \quad Y_A = G_1 + G_2 = 2G,$$

$$\sum m_A(P_k) = G_1 l \sin\alpha + P_1^{\text{in}}(m - l\cos\alpha) + X_B(m+n)$$
$$- G_2 l \sin\alpha - P_2^{\text{in}}(m + l\cos\alpha) = 0.$$

From this last equation we find

$$X_A = X_B = \frac{2P^{\mathrm{in}} l \cos\alpha}{m+n} = \frac{Gl^2 \sin 2\alpha}{g(m+n)}\,\omega^2. \qquad (1)$$

If the cross-beam did not rotate ($\omega = 0$), then the horizontal constraint reactions, X_A and X_B, would be zero and, hence, the bearing and thrust bearing would not experience lateral pressure. But when the cross-beam is in rotation, these reactions (and, hence, the lateral pressures on the constraints) are proportional to the square of the angular velocity of the cross-beam and may attain very considerable magnitudes. Since the plane in which the cross-beam lies turns together with the cross-beam as it rotates, the dynamic reactions that arise generate a beating of the shaft AB in the bearings and increase their wear.

The reactions X_A and X_B may be equal to zero even when the shaft is in rotation ($\omega \neq 0$) but, as is evident from (I), this requires that $\sin 2\alpha = 0$, which is only possible for $\alpha = 0$ and $\alpha = 90°$, that is, when the axis of rotation is the axis of symmetry of the rotating system.

A rotating body whose reactions in the bearings do not depend on the rate of rotation of the body is said to be dynamically balanced.

For a rotating body to be dynamically balanced, it is often given the shape of a solid of rotation so that it rotates on its axis of symmetry.

A nonsymmetric body can also be dynamically balanced. But, as is shown in more detailed courses of mechanics, this requires that the axis of rotation of the body be one of the so-called principal central axes of inertia of the given body. If due to unavoidable errors of manufacture the axis of rotation of the body is not perfectly coincident with the axis of symmetry (in the general case, with its principal central axis of inertia), then these errors are eliminated by special procedures (by a so-called dynamic balancing of the body).

CHAPTER XIX

THE WORK AND POWER OF A FORCE

Sec. 87. The work of a constant force over a rectilinear path

If under the action of a given force the magnitude of the velocity of a particle, to which the force is applied, varies as the particle is displaced, the force performs work. The greater the magnitude of the force and the path length covered by the point of application of the force, the greater the amount of work done. In the most elementary case, when the line of action of the force coincides with the direction of motion of the point of its application (for example, the line of action of the force of gravity of a body moving vertically), the work A of the force is equal to the product of its magnitude P into the path length S covered by the point of its application.

If in the process the direction of the force coincides with the direction of motion of the point of its application (for example, the direction of the force of gravity of a body falling vertically), then the force produces an acceleration and the work of the force is regarded as being positive, $A = PS$. But if the direction of the force is opposite the direction of motion of the point of its application (for example, the direction of the force of gravity of a body thrown vertically upwards), then the force causes deceleration of motion. In this case, the work of the force is said to be negative, $A = -PS$. And finally, when the direction of the force is perpendicular to the direction of displacement of the point of its application (for instance the direction of the force of gravity of a body sliding along a horizontal plane), then this force does not affect* the magnitude (absolute value) of the velocity of the point. For this reason the *work of a force perpendicular to the direction of motion of the point of its application is zero*: $A = 0$.

---

* A force directed along the normal to the path of motion of a point imparts to the point a normal acceleration which can only change the direction of motion of the point (if this is not prevented by constraints imposed on the point) or change the pressure on the constraint.

Now let us consider a more general case.

Suppose a point M of a body to which a force **P** (constant in magnitude and direction) is applied* is in straight-line motion from position M to position M' (Figs. 222*a* and 222*b*), the angle between the direction of the force and the direction of displacement of the point being α and the path covered by the point being S.

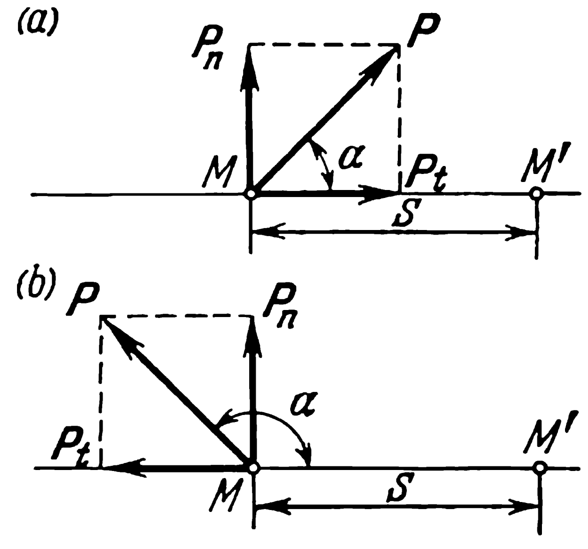


FIG. 222

The force **P** may be decomposed into two components: a normal component $\mathbf{P}_\mathrm{n}$ that does not perform any work, and a tangential component $\mathbf{P}_\mathrm{t}$ whose magnitude $P_\mathrm{t} = P \cos \alpha$.

Since only the second component performs work, the work of force **P** is

$$A = P_\mathrm{t}S = PS \cos \alpha. \quad (138)$$

The work of a constant force in the case of rectilinear displacement of the point of its application is equal to the product of the magnitude of the force by the path length covered by that point and by the cosine of the angle between the direction of the force and the direction of the motion of its point of application.

The work of a force is a scalar quantity, that is, it is fully determined by its numerical value and its sign.

It will readily be seen (Fig. 222*a*) that if the angle α is acute, the direction of the tangential component $\mathbf{P}_\mathrm{t}$ of the force **P** will coincide with the direction of motion of the point of its application. The motion of the point will be accelerated and, by formula (138), the work of the force will be positive. *Forces whose direction form an acute angle with the direction of motion of their point of application perform positive work.*

---

* Note that when we speak of the work of a given force applied to some point of a body, we do not insist that the body is acted upon by this force alone or that it is the resultant of all forces impressed on the body. Other forces may be acting on it at the same time.

Now if the angle α is obtuse (Fig. 222*b*), the direction of the tangential component $\mathbf{P}_t$ of the force $\mathbf{P}$ is in the opposite direction to the motion of its point of application. The motion of this point is decelerated and (since the cosine of an obtuse angle is a negative quantity) the work of the force is negative. *Forces whose direction form an obtuse angle with the direction of motion of their point of application perform negative work.*

If the direction of a constant force coincides with the direction of motion of the point of its application or is directly opposite to the motion of the latter, then, as is evident from formula (138),

$$A = \pm PS. \tag{139}$$

This formula is applicable not only in the case of rectilinear motion of the point of application of a constant force, but also for curvilinear motion, provided that the force is all the time directed along the tangent to the path of motion of its points of application.

The work of a force clearly has the following dimensions:

$$[A] = [\text{force}] \times [\text{length}]$$

*The unit of work in the SI is the work of a force of 1 N when the body is displaced a distance of 1 meter in the direction of the action of the force. This unit is termed the Joule** (abbreviated J).

In the engineering system of units, work is measured in kilogram-force·metre (abbreviated kgf·m). Since a force of 1 kgf = 9.81 N, it follows that

$$1 \text{ kgf}\cdot\text{m} = 1 \text{ kgf} \times 1 \text{ m} = 9.81 \text{ N} \times 1 \text{ m} = 9.81 \text{ J}.$$

The concept of work that is given in mechanics (sometimes called mechanical work) arose out of everyday experience. However, one should bear in mind that it does not always coincide with what is understood by work from the physiological point of view. For instance, a person standing still and holding a heavy load with extended arms does not, quite obviously, perform any mechanical work ($S = 0$), but from the physiological standpoint he of course performs a definite amount of work (which may be quite considerable if the load is heavy).

---

* In honour of the outstanding English physicist J. P. Joule (1818-1889).

Problem 98. Determine the work that must be performed in order to move a body of mass $m = 10$ kg over a distance $S = 10$ m along a horizontal floor. The coefficient of friction is $f = 0.5$.

Solution. In this case the force of gravity of the body does not perform any work because its direction is perpendicular to that of the motion of the body. For uniform displacement along a horizontal floor, we have to apply a horizontal force **P** equal in magnitude to the force of friction **F** between the body and the floor:

$$P = F = fR_n = fG.$$

The body weighs $G = mg = 10 \times 9.81$ kg·m/s^2 = 981 N. Hence, $P = F = 0.5 \times 981 = 490$ N, whence $A = PS = 490 \times 10 = 4910$ N · m = 4.9 kJ (kilojoules).

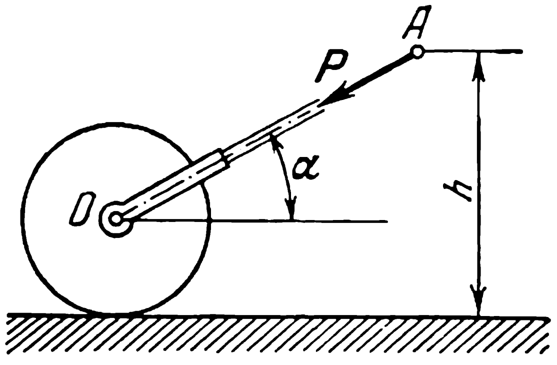


FIG. 223

Problem 99. Referring to Fig. 223, a cylindrical roller of diameter $d = 60$ cm is set into motion by a man pressing on a handle AO with a constant force $P = 120$ N in the direction AO; $AO = 1.5$ m, the point A is $h = 1.2$ m above the horizon. Find the work performed in displacing the roller a distance $S = 5$ m.

Solution. Since the force **P** is directed at a certain angle α to the direction of the displacement of point O (point of application), the work of the force is $A = PS \cos \alpha$.

In Fig. 223 we have $\sin \alpha = \frac{h - d/2}{AO} = \frac{1.2 - 0.3}{1.5} = 0.6$. Hence, $\cos \alpha = \sqrt{1 - \sin^2 \alpha} = 0.8$ and $A = 120 \times 5 \times 0.8 = 480$ J.

Sec. 88. The work of a variable force over a curvilinear path. A graphical depiction of the work

Using the concept of the work of a constant force over a rectilinear path that was described in the preceding section, we now calculate the work of a force in the most general case.

Referring to Fig. 224, suppose the point of application M of a variable (in magnitude and direction) force **P** is displaced from position A to position B and describes a certain curvilinear path. First we partition the path $\overset{\frown}{AB} = S$ traversed by the point into a large number n of small portions such that each portion may be assumed to be straight (to a high degree of accuracy) and the force acting over such a portion may be regarded as constant both in magnitude and in direction.

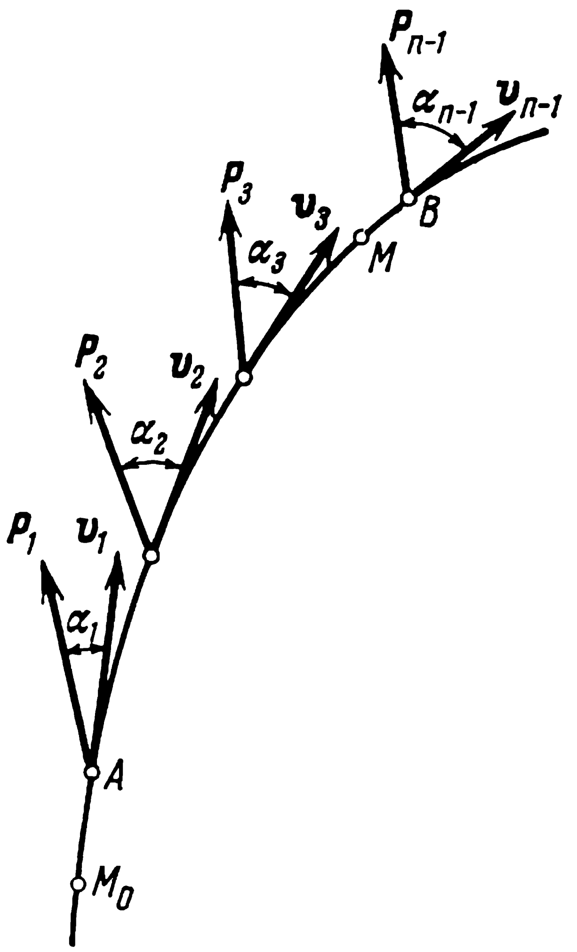


FIG. 224

Denote by $P_1, P_2, P_3, \ldots, P_{n-1}$ the constant magnitudes of the variable force **P** over the specified portions, by ΔS_1, ΔS_2, ΔS_3, . . ., ΔS_{n-1} the lengths of the corresponding (straight-line) portions, and by α_1, α_2, α_3, . . ., α_{n-1} the angles between the corresponding directions of the force and the velocity of the point of application of the force.

On the basis of the foregoing [formula (138)], the work of the force $\mathbf{P}_1$ over the portion ΔS_1 is $\Delta A_1 = P_1 \Delta S_1 \cos \alpha_1$. The work of the force over other portions of the curvilinear path is found in the same way.

The total work A of the variable force **P** over the finite path AB is, clearly, equal to the sum of the works over all its portions:

$$A = \Delta A_1 + \Delta A_2 + \ldots + \Delta A_{n-1} = P_1 \Delta S_1 \cos \alpha_1 + P_2 \Delta S_2 \cos \alpha_2 + \ldots + P_{n-1} \Delta S_{n-1} \cos \alpha_{n-1}.$$

Using the symbol for an algebraic sum of homogeneous terms, we can write

$$A = \sum P_k \Delta S_k \cos \alpha_k.$$

It is quite obvious that the larger the number of portions n into which the path covered by the point of application of the variable force **P** is subdivided, the more exactly is the work of the force over the path calculated. In the limit, when the number of portions n becomes infinitely great, the length of each of them becomes infinitely small.

The work of a force over an infinitely small displacement of its point of application is termed the elementary work.

Denoting the elementary work by dA and the length of an infinitesimal element of path by dS, we have

$$dA = P\,dS \cos \alpha. \tag{140}$$

The elementary work of a force is equal to the product of the magnitude of the force at a given time by the length of the path element and the cosine of the angle between the direction of the force and the direction of the displacement of the point of its application at that time.

Finding the expression for the elementary work of a force over each path element dS, we can determine the work performed over the whole finite path $\overset{\frown}{AB} = S$ as the limit of a sum of an infinitely great number of infinitesimal terms, which is to say, as the integral of the elementary work computed at the limits of the range of the point of application of the force (from zero to S):

$$A = \int_A^B dA = \int_0^S P \cos \alpha \times dS. \tag{141}$$

The work of a variable force over a finite path is equal to the integral of the elementary work of the given force, which integral is computed at the limits of the range of the point of application of the force.

To compute the work of a variable (in magnitude and direction) force **P** by formula (141), the variable P and variable $\cos \alpha$ that appear under the sign of the integral must be expressed as functions of the path S.

How to evaluate this integral in certain special cases will be shown in Secs. 90 and *91. For the present, noting that calculating the given integral is very difficult in many cases, let us examine a simpler method (graphical) that is frequently encountered in engineering practice in calculating the work of a variable force.

Let the point M of application of a variable (in magnitude and direction) force **P** be displaced from position M_1 to position M_2, which are given on the trajectory of the force by the distances $s_1 = \overset{\frown}{OM_1}$ and $s_2 = \overset{\frown}{OM_2}$ reckoned from some origin O (Fig. 225).

In a rectangular coordinate system (Fig. 226) lay off to some chosen scale the following quantities: the distance s from the origin on the axis of abscissas and, on the axis of ordinates, the corresponding value of the projection of the force **P** on the direction of the velocity of the point of its application M, that is, the algebraic value of the tangential component of the given force $P_t = P \cos \alpha$.

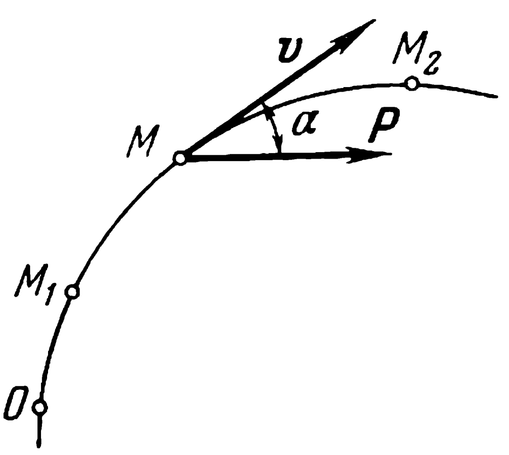


FIG. 225

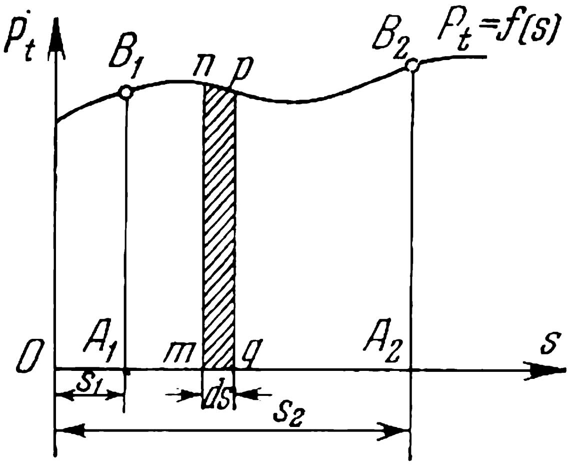


FIG. 226

Joining the points having the given coordinates of s and P_t with a continuous curve, we obtain a graph of the relationship $P_t = f(s)$.

The work of the force **P** *over the path length S is then depicted (to a certain scale) by the area of the figure* $A_1B_1B_2A_2$ (Fig. 226) *bounded by the axis of abscissas, the curve* $P_t = f(s)$ *and by two ordinates corresponding to the initial and terminal positions of the point of application of the force* **P**.

Indeed, the area of some elementary rectangle $mnpq$ (cross-hatched in Fig. 226), the base of which $mq = dS$ and the height $mn = P_t = P \cos \alpha$, is the magnitude of the elementary work of the given force:

$$dA = P\, dS \cos \alpha = P_t\, dS = mn \times mq.$$

Now the area of the figure $A_1B_1B_2A_2$ is the limit of the sum of an infinite number of such elementary rectangles, that is,

$$\text{area } A_1B_1B_2A_2 = \int_{S_1}^{S_2} P\cos\alpha \times dS.$$

In Fig. 226 the values of the tangential component of the force, P_t, are laid off upwards (in the positive direction of the axis), and therefore the work of the force **P** given by the area of the figure $A_1B_1B_2A_2$ is positive. If the constructed curve $P_t = f(s)$ or a portion of it is located in the direction of negative values of P_t from the axis of abscissas, then the appropriate area will depict negative work.

When computing the work of a force by the graphical method, one of course has to take into account the scales in which the distances s and the corresponding magnitudes of the force P_t are laid off on the graph $P_t = f(s)$. Suppose we took the following scales: μ_s m/mm for the distances and μ_P N/mm for the force. Then, if the area of the figure $A_1B_1B_2A_2$ is equal to F mm^2, the work of the force will be $A = F\mu_s\mu_P$ J.

The areas of curvilinear figures may be computed by a number of methods and also by a device called a planimeter.

The graphical method is employed to determine the work of a force when we know the values of the force P only for separate values of s, and it is either difficult or impossible to establish the analytic relationship $P_t = f(s)$. In certain cases (for instance, when determining the work of steam or gas in the cylinders of a steam engine or other engine), the graph of the relation $P_t = f(s)$ is obtained automatically by recording instruments called indicators.

Sec. 89. A theorem on the work of a resultant

Theorem. *The work of the resultant of several forces over some path is equal to the algebraic sum of the works of the component forces over that path.*

Referring to Fig. 227, suppose there are several forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ applied to a particle M that is moving from position M_1 to position M_2 along any curvilinear path. Denote the resultant of the given forces by $\mathbf{R}_P$. As we know, the projection of the resultant on any axis is equal to the algebraic sum of the projections of the components on that axis. Taking for the projection axis the axis having the direction of the velocity $\mathbf{v}$ of the point of application of the forces and denoting the angles between the forces $\mathbf{R}_P, \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ and the axis of projections (the direction of the vector $\mathbf{v}$) by $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$, respectively, we get

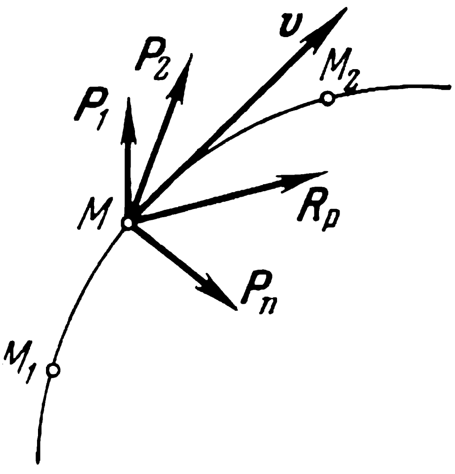


FIG. 227

$$R_P \cos \alpha = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots + P_n \cos \alpha_n.$$

Multiplying both sides of this equation by an infinitesimal element of path dS, we find that the *elementary work of the resultant is equal to the algebraic sum of the elementary works of the component forces*:

$$R_P\, dS \cos \alpha = P_1\, dS \cos \alpha_1 + P_2\, dS \cos \alpha_2 \ldots + P_n\, dS \cos \alpha_n.$$

Integrating this equation from zero to S (the range of the point M along the trajectory $\overset{\frown}{M_1M_2}$), we find

$$\int_0^S R_P \cos \alpha \times dS = \int_0^S P_1 \cos \alpha_1 \times dS + \int_0^S P_2 \cos \alpha_2 \times dS + \ldots + \int_0^S P_n \cos \alpha_n \times dS.$$

From this, since the integrals in both members of this equation are expressions of the work of the corresponding *forces over a given finite path S of their point of application, we obtain*

$$A_{R_P} = A_{P_1} + A_{P_2} + \dots + A_{P_n}$$

which completes the proof of the theorem.

Sec. 90. The work of the force of gravity

Replace the forces of gravity of all particles of a body by a single resultant force G that is equal in magnitude to the weight of the body. When the body is in motion, suppose the centre of gravity, C, moves along a certain curvilinear path (in the vertical plane xOy of Fig. 228) from position C_1 to position C_2. Denote the ordinates of points C_1 and C_2 by y_1 and y_2 respectively.

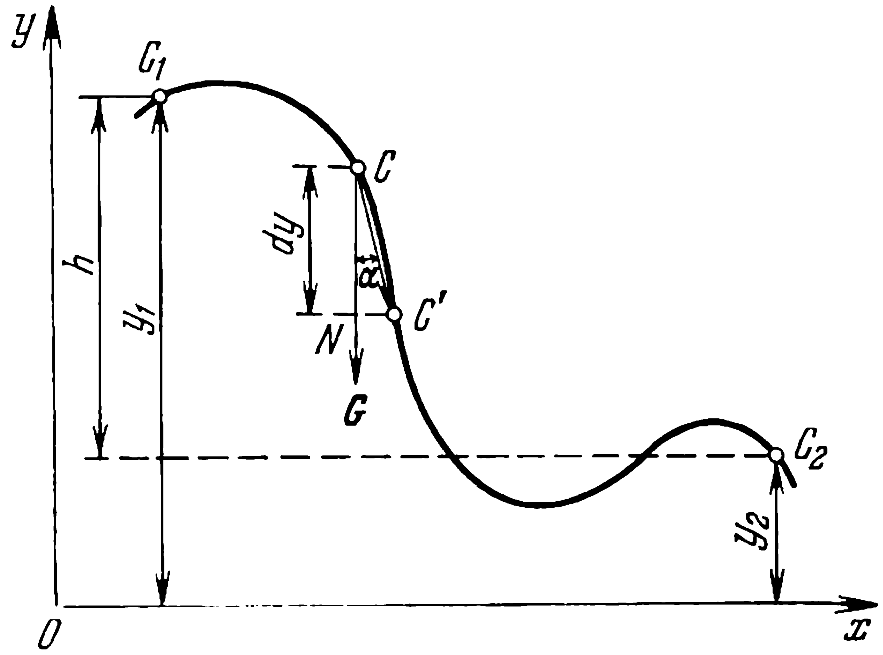


FIG. 228

Select some infinitesimal portion CC' of the path and find the elementary work of the constant (in magnitude and direction) force G over that infinitesimal displacement dS of its point of application.

By the formula (140), $dA = G\,dS\cos\alpha$, where α is the angle between the direction of the elementary displacement $\overrightarrow{CC'}$ of point C and the direction of the force G. The quantity $dS\cos\alpha$ may be regarded as the projection of the vector of the elementary displacement $\overrightarrow{CC'}$ on the y-axis. Since this projection is directed towards the negative y-axis, it follows that $dS \cos \alpha = CN = -dy$ (the point C moves towards diminishing ordinates). Thus, the elementary work of the force of gravity is

$$dA = -G\,dy.$$

And the work of this force over the finite path $\overset{\frown}{C_1C_2}$ is given by

$$A = \int_{C_1}^{C_2} dA = \int_{y_1}^{y_2} (-G)\,dy = -G \int_{y_1}^{y_2} dy$$

$$= -G(y_2 - y_1) = G(y_1 - y_2).$$

Denoting by h the vertical displacement of the centre of gravity of the body, that is, the difference between the initial y_1 and terminal y_2 heights above some horizontal plane, we get

$$A = Gh. \tag{142}$$

The work of the force of gravity of a body is equal to the product of the weight of the body by the magnitude of the vertical displacement of its centre of gravity. If, over the path length under consideration, the centre of gravity of the body drops ($y_1 > y_2$ and the difference $h = y_1 - y_2$ is positive), then the work of the force of gravity is positive. But if the centre of gravity of the body rises ($y_1 < y_2$ and $h = y_1 - y_2$ is negative), then the work of its centre of gravity is negative.

If, over the path length under consideration, the terminal height y_2 of the centre of gravity of the body were the same as its initial height y_1, then the variation h in height of the centre of gravity of the body and, hence, the work of the force of gravity for the given displacement of the body would be equal to zero.

Hence the work of the force of gravity of a body depends only on its weight and on the change in the height of its centre of gravity, and does not depend on the shape of the trajectory of the centre of gravity of the body and the path length traversed by that point.

Problem 100. Referring to Fig. 229, a load weighing $G = 500$ N is uniformly displaced to a height $h = 5$ m along a plane inclined to the horizontal at an angle $\alpha = 30°$. Find the work performed if the coefficient of sliding friction is $f = 0.4$ and if a force **P** parallel to the inclined plane is applied to the load.

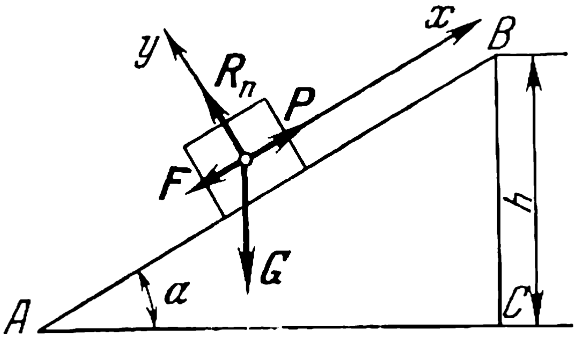


FIG. 229

Solution. For a uniform displacement of the load along the plane, the force **P** applied to it must balance the action on the load of all other applied forces: the force of gravity **G**, the normal reaction of the plane $\mathbf{R}_n$, the force of friction **F**.

We will work the problem by two different methods.

1. Drawing the coordinate axes x and y as shown in Fig. 229, set up the equilibrium equations of the forces applied to the load:

$$\sum Y_k = R_n - G\cos\alpha = 0, \text{ whence } R_n = G\cos\alpha,$$

$$\sum X_k = P - G\sin\alpha - F = 0, \text{ whence } P = G\sin\alpha + F.$$

However, the magnitude of the force of sliding friction in the case of motion is $F = fR_n = fG\cos\alpha$. Thus, the magnitude of the force needed for uniform displacement of the load is

$$P = G\sin\alpha + F = G(\sin\alpha + f\cos\alpha).$$

From the triangle ABC we find the path $S = AB$ traversed by the load as it is raised to a height $h = BC$:

$$\sin\alpha = \frac{BC}{AB} = \frac{h}{S}, \quad \text{whence} \quad S = \frac{h}{\sin\alpha}.$$

Since the direction of the force **P** coincides with the direction of displacement, the desired work is

$$A = PS = \frac{G(\sin\alpha + f\cos\alpha)h}{\sin\alpha} = \frac{500(0.5 + 0.4 \times 0.866) \times 5}{0.5}$$

$$= 4232 \text{ N}\cdot\text{m} \approx 4.23 \text{ kJ}.$$

2. The work A' of the resultant of the forces **G**, $\mathbf{R}_n$ and **F** is equal to the algebraic sum of the works of all the component forces.

The work of the force of gravity in raising the load to height h is $A_1 = -Gh$, The work A_2 of the normal reaction $\mathbf{R}_n$ in displacing the load along the plane is always zero because the force $\mathbf{R}_n$ is perpendicular to the displacement.

•The work A_3 of the force of friction **F**, which is always in the opposite direction to that of the motion, is

$$A_3 = -FS = -fG\frac{h}{\sin\alpha}\cos\alpha$$

Consequently, the work of the resultant is

$$A' = A_1 + A_2 + A_3 = -Gh - fG\frac{h}{\sin\alpha}\cos\alpha = -\frac{G(\sin\alpha + f\cos\alpha)h}{\sin\alpha}$$

Now from statics we know that the balancing force **P** of a given system of forces and the resultant of that system are always equal in magnitude and opposite in direction along one straight line, whence the desired work A of the force **P** is equal to the work A' of the resultant taken with opposite sign:

$$A = -A' = 4.23 \text{ kJ}.$$

Sec. *91. The work of an elastic force

Take a helical spring and fix the upper end. If we ignore the weight of the spring, we may regard it as being in an undeformed state (Fig. 230*a*). Now (Fig. 230*b*) stretch the spring by applying a vertical downward force **P'** to the free end M. Within certain limits, there will be a direct relationship between the force stretching the spring and its deformation (elongation). The greater the elongation of the spring, the greater the force required to retain it in its extended state. This is due to the fact that when a real body is deformed, there always arise, as we have already mentioned, certain inner forces, called forces of elasticity, that resist the action of external forces on the body. As a result, the spring will act on the point M with a certain force **P** that will balance the external applied force **P'**. Since these forces are equal in the case of equilibrium, the force **P** will also be proportional to the elongation of the spring. The force **P** is termed an *elastic reaction* or *elastic force*. Denoting the elongation of the spring by x (Fig. 230*b*), we will have

$$P = cx \tag{143}$$

where c is a proportionality factor that is constant for a given spring; it is called the ***stiffness coefficient****[*] of the spring.

Referring to Fig. 230c, let us compute the work of the variable force **P** when the spring is stretched by $M_1M_2 = h$. Take the rectilinear trajectory of point M for the x-axis (vertically downwards in Fig. 230b). For the coordinate origin take O, which corresponds to the position of M in the undeformed state of the spring.

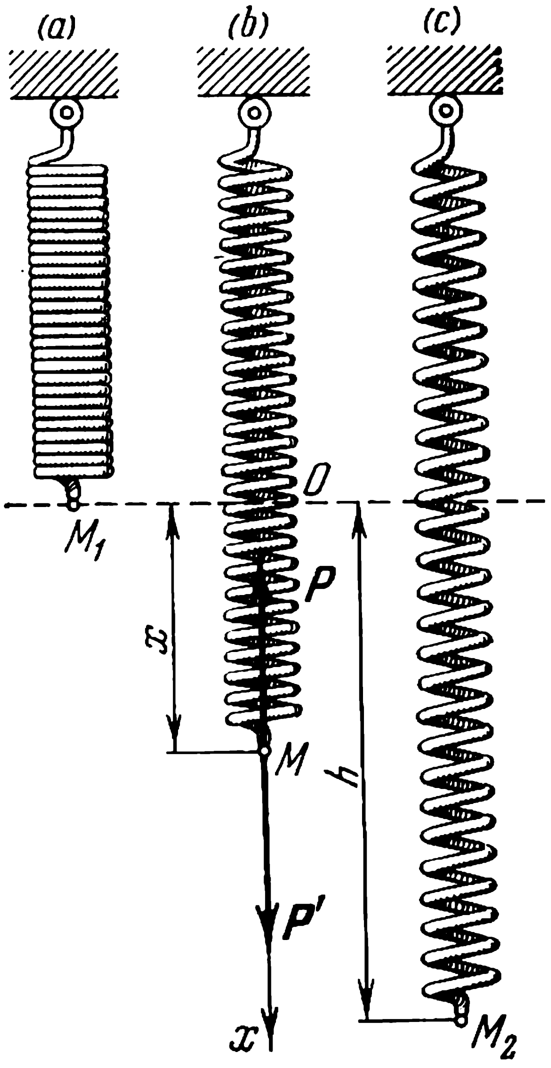


FIG. 230

By formula (140),

$$dA = P\,dS \cos \alpha.$$

Taking into account that $P = cx$, $dS = dx$ and $\cos \alpha = \cos 180° = -1$ (since the elastic force **P** of the spring is in the opposite direction to the rectilinear displacement of its point of application M), we find

$$dA = -cx\,dx.$$

Now the work of the force **P** performed over the finite displacement $M_1M_2 = h$ of its point of application is equal to

$$A = \int_{M_1}^{M_2} dA = \int_0^h (-cx)\,dx = -\frac{cx^2}{2}\Big|_0^h = -\frac{ch^2}{2}.$$

---

[*] The stiffness coefficient c of a spring may be computed from a formula derived in the course of strength of materials. The value of the coefficient for a given spring may be found from a simple experiment. Suspend a load from the spring and determine its so-called static elongation Δl_{st}, that is, the elongation of the spring in the position of equilibrium of the load (when oscillations cease). In this position, the elastic force of the spring is equal, in magnitude, to the weight G of the suspended load. Therefore,

$$c = \frac{P}{x} = \frac{G}{\Delta l_{st}}.$$

The same formula is used to determine the work not only of the elastic force of a stretched spring but also the work of the elastic force that appears in the tension and compression of any rectilinear rod, the bending of a beam and so forth—in all cases where the magnitude of the elastic force is determined from the formula $P = cx$, where c is a certain (consant for each body) proportionality factor and x is a value (that corresponds to the given value of the force) of the displacement of the point of application of the force reckoned from the position of that point in the undeformed state of the body.

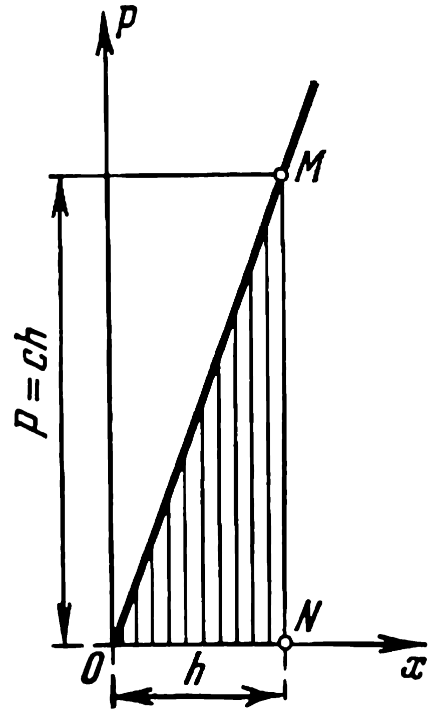


FIG. 231

Thus, *the work of the elastic force* **P** *of a body, as expressed by the law* $P = cx$, *is numerically equal to half the product of the proportionality factor in the expression of the given force by the square of the displacement of its point of application as reckoned from the position of that point in the undeformed state of the body*:

$$A = -\frac{ch^2}{2}. \qquad (144)$$

The minus sign in (144) shows that the work of the elastic force, like any force of resistance, is negative. Since the external stretching force is equal in magnitude to the elastic force and is in the opposite direction, the work A' of the external force is equal to A in absolute value but is positive:

$$A' = \frac{ch^2}{2}.$$

Such is the work that has to be done in order to elongate a spring by the amount h.

The relationship between the numerical value of the elastic force $P = cx$ and the displacement x of the point of its application is obviously expressed graphically as a straight line (Fig. 231).

In accordance with what was said earlier (see Sec. 88), the work of this force over a path h is expressed (to a certain scale) as the area of the triangle ONM. In this way, we again arrive at the above-established numerical value of the work of an elastic force:

$$A=\frac{MN\times ON}{2}=\frac{ch\times h}{2}=\frac{ch^2}{2}.$$

Sec. 92. Power. Efficiency

The power of a force is a quantity that characterizes the rate at which the given force performs work at a given time.

The average power N_{av} of a force over a certain time interval t is equal to the ratio of the work A performed in that time to the given time interval:

$$N_{\text{av}}=\frac{A}{t}. \tag{145}$$

If the work is performed uniformly, that is, the same amounts in equal and arbitrarily taken time intervals, *then the true power N of the force is measured by the work performed by it in unit time*, and coincides, therefore, with the average power:

$$N=\frac{A}{t} \tag{146}$$

which in this case is constant.

If the work is performed in a nonuniform fashion, then the average power is a variable quantity that depends on the time interval taken. In that case, use is made of the concept of power at a given instant of time.

The power N of a force at a given time t is equal to the ratio of the elementary work dA of the force during an infinitesimal time interval, beginning at time t, to the value dt of that time interval:

$$N=\frac{dA}{dt}. \tag{147}$$

It is clear that the dimensions of power are:

$$[N]=\frac{[\text{work}]}{[\text{time}]}.$$

The unit of power in the International System (SI) is the power such that 1 joule of work is performed in 1 second. This unit of power is termed the watt (abbreviated W)*:

$$1 \text{ W} = 1 \text{ J/s}.$$

In the engineering system of units, the unit of power is the power generated by the work of 1 kgf·m in 1 second. This unit is called the kilogram-force-metre per second (abbreviated 1 kgf·m/s). A bigger unit of power termed the horsepower (abbreviated hp) is sometimes used:

$$1 \text{ hp} = 75 \text{ kgf·m/s}.$$

Now let us find the relationship between a horsepower and a watt. Above (Sec. 87) we established that 1 kgf·m = = 9.81 J. Consequently,

$$1 \text{ hp} = 75 \text{ kgf·m/s} = 75 \times 9.81 \text{ J/s} = 736 \text{ W} = 0.736 \text{ kW}. \quad (148)$$

The units of power called watt, kilowatt and so forth should not be confused with the electrical units of work: watt-hour, kilowatt-hour and so forth. The watt-hour is the work performed in one hour by a force having the power of one watt. Thus, 1 W·h = 1 J/s × 3600 s = 3600 J. The formula (147) of power at a given time may be cast in another form if we substitute into it the earlier established [formula (140)] expression of elementary work $dA = P\,dS \cos\alpha$:

$$N = \frac{dA}{dt} = \frac{P\,dS\cos\alpha}{dt}.$$

Since the derivative dS/dt of the displacement of the point of application of the force with respect to time is equal to the magnitude of the velocity v of the point, it follows that

$$N = Pv\cos\alpha. \quad (149)$$

The power of a force at a given time is equal to the product of the magnitude of the force, the magnitude of the velocity of its point of application, and the cosine of the angle between the

---

* James Watt (1736-1819), outstanding Scottish engineer and inventor.

directions of the force and the velocity of its point of application, all corresponding to the given time.

If the direction of the force coincides with the direction of the velocity of its point of application, then $\alpha = 0$ and $\cos\alpha = 1$. In this case, formula (149) is simplified to

$$N = Pv. \tag{150}$$

Expressing the force in newtons and the velocity in metres per second, we obtain the power in watts in formulas (149) and (150).

In the operation of any machine, part of the power consumed is used to overcome unwanted resistances that unavoidably occur in machine operation. For example, the power consumed by a lathe is used not only to perform useful work (machining) but also to overcome the friction in the moving parts of the machine and to overcome air resistance.

The ratio of the useful power N_{use} *of a machine to the consumed power* N *or the ratio of the useful work* A_{use} *over a certain time interval to the entire work* A *during that time interval is termed the mechanical efficiency.*

Denoting the efficiency as usual by η (the Greek letter eta), we have

$$\eta = \frac{N_{\text{use}}}{N} = \frac{A_{\text{use}}}{A}. \tag{151}$$

Efficiency is one of the most important characteristics of a machine. It shows rational utilization of power. As is evident from formula (151), the higher the efficiency of a machine, the greater the useful power N_{use} in the power consumption, and the smaller the losses of power due to various useless resistances.

The efficiency of a machine is always less than unity because there is no way to dispense completely with all useless resistances.

Problem 101. A turbine with efficiency $\eta = 0.8$ is installed to utilize the work of a waterfall. Determine the useful power of the turbine in kilowatts if the waterfall yields 600 m^3 of water every minute from a height of 6 m.

Solution. The mass of 1 cubic metre of water is equal to 1 Mg (megagram)*. The weight of water falling in one minute is $G = mg = 600 \times 10^3\ \text{kg} \times 9.81\ \text{m/s}^2 = 5886 \times 10^3\ \text{N} = 5886\ \text{kN}$.

---

* $1\ \text{Mg} = 10^6\ \text{g} = 10^3\ \text{kg}$.

By formula (146), the power consumed by the turbine is

$$N=\frac{A}{t}=\frac{Gh}{t}=\frac{5886\times 6}{60}\,\frac{\text{kN}\cdot\text{m}}{\text{s}}=589\ \text{kW}.$$

By formula (151), the useful power of the turbine is

$$N_{\text{use}}=\eta N=0.8\times 588.6\approx 471\ \text{kW}.$$

Problem 102. What must the diameter be of the piston of a one-cylinder steam engine with the average steam pressure on the piston $p = 400$ kPa, average velocity of the piston $v = 2$ m/s, useful power of the machine $N_{\text{use}} = 50$ kW, and its mechanical efficiency $\eta = 0.9$?

Solution. The power consumed by the machine is found from formula (150):

$$N=\frac{N_{\text{use}}}{\eta}=\frac{50}{0.9}\approx 55.6\ \text{kW}.$$

Since the direction of the force of pressure of the steam on the piston coincides with the direction of motion of the piston, we have $\cos\alpha = 1$ and by formula (150) the power is $N = Pv$.

From this we find that the force with which the steam acts on the piston is

$$P=\frac{N}{v}=\frac{55.6}{2}=27.8\ \text{kN}.$$

On the other hand, this force is equal to the pressure of the steam, p, multiplied by the area of the piston $F = \pi d^2/4$:

$$P=pF=p\,\frac{\pi d^2}{4}.$$

Hence the diameter of the piston must be

$$d=\sqrt{\frac{4P}{\pi p}}=\sqrt{\frac{4\times 27.8}{3.14\times 400}}\approx 0.297\ \text{m}=29.7\ \text{cm}.$$

Sec. 93. The work and power of a force applied to a rigid body rotating about a fixed axis

Referring to Fig. 232, suppose a force **P** is applied to a certain point M of a rigid body rotating about a fixed axis z. Resolve this force into two mutually perpendicular components: **P**′ lying in the plane Pl perpendicular to the z-axis of rotation of the body, and **P**″ perpendicular to this plane, that is, parallel to the z-axis.

By the theorem on the work of a resultant, the elementary *work of the force* **P** *is equal to the sum of the elementary* works of the component forces **P**′ and **P**″. But the force **P**″ is perpendicular to the displacement of point M in plane Pl and so its work is equal to zero. Thus, the elementary work dA of the force **P** is equal to the elementary work of the component **P**′:

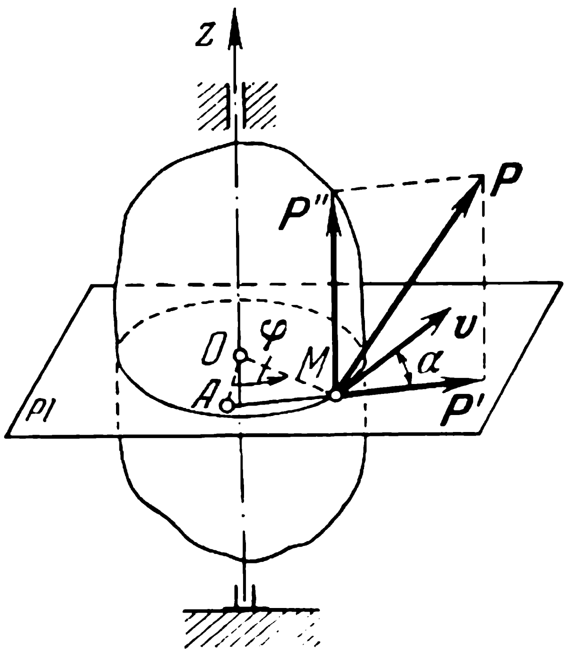


FIG. 232

$$dA = P'\,dS\cos\alpha.$$

As always in the expression of work, here $\cos\alpha = \cos\widehat{(\mathbf{P}', \mathbf{v})}$.

The trajectory of a point M of a rigid body rotating about a fixed axis is a circle of radius $OM = r$ and, consequently, the elementary displacement of this point $dS = r\,d\varphi$, where $d\varphi$ is an elementary angle of rotation of the body (in radians). (It is assumed that the elementary rotation is in the positive sense of reckoning angles φ, as depicted in Fig. 232.) Hence,

$$dA = P'r\cos\alpha \times d\varphi.$$

Recalling the concept (established in statics in Sec. 40) of the moment of a force with respect to an axis, we find the moment of the force **P** about the axis of rotation z:

$$m_z(\mathbf{P}) = m_O(\mathbf{P}') = P' \times OA,$$

where **P**′ is the projection of force **P** on the plane Pl perpendicular to the z-axis, and OA is the length of the perpendicular dropped from the point O of intersection of the z-axis with the plane Pl on the line of action of the force **P**′.

The sides of the angle AOM are perpendicular to the vectors **P**′ and **v** and therefore

$$\angle AOM = \widehat{(\mathbf{P}', \mathbf{v})} = \alpha;$$

hence,

$$OA = OM \times \cos\alpha = r\cos\alpha.$$

The moment of the force about the axis of rotation of a body is frequently called the *rotational* (or *turning*) *moment* (*torque*).

Denoting this moment by M_{rot}, we find that $M_{\mathrm{rot}} = m_z(\mathbf{P}) = P'r\cos\alpha$. Making the appropriate substitution in the above-found expression for the elementary work of a force **P**, we finally obtain

$$dA = M_{\mathrm{rot}}\, d\varphi. \tag{152}$$

The elementary work dA of a force applied to a rotating body is equal to the product of the rotational moment M_{rot} of the force by the elementary angle $d\varphi$ of rotation of the body.

The entire conclusion is made on the assumption that the elementary rotation occurs in the positive direction and the rotational moment is also positive. It is easy to see that formula (152) holds true for any signs of M_{rot} and $d\varphi$: if the signs are the same, the work is positive, if the signs are different, it is negative.

The work of a force during rotation of the body through a finite angle φ is equal to

$$A = \int_0^{\varphi} M_{\mathrm{rot}}\, d\varphi. \tag{153}$$

If the rotational moment M_{rot} is a variable quantity, then to compute this integral we must know the dependence of the rotational moment on the angle φ:

$$M_{\mathrm{rot}} = f(\varphi).$$

But when M_{rot} = constant, we have

$$A = \int_0^{\varphi} M_{\mathrm{rot}}\, d\varphi = M_{\mathrm{rot}} \int_0^{\varphi} d\varphi = M_{\mathrm{rot}}\, \varphi.$$

For a constant rotational moment M_{rot}, the work A is equal to the product of the moment by the angle of rotation φ of the body:

$$A = M_{\mathrm{rot}}\, \varphi. \tag{154}$$

Now let us find the power of the force* applied to a rotating body by substituting into (147) the appropriate expression (152) of elementary work:

$$N=\frac{dA}{dt}=\frac{M_{\text{rot}}\,d\varphi}{dt}.$$

But $d\varphi/dt=\omega$, where ω is the angular velocity of the body. Hence

$$N=M_{\text{rot}}\omega. \tag{155}$$

The power N of a force applied to a rotating body is equal to the product of the rotational moment M_{rot} of the force by the angular velocity ω of the body.

If the rotational moment M_{rot} is expressed in N·m (newton-metres) and the angular velocity ω is given in rad/s, then the power defined by formula (155) is expressed in watts.

If the force **P** applied to a rotating body is directed along a tangent to the circle described by the point of its application, it is frequently termed a peripheral force.

In this case, clearly,

$$M_{\text{rot}}=Pr, \tag{156}$$

where P is the peripheral force and r is the radius of the circle described by the point of application of the force.

Problem 103. A driving pulley of diameter 0.4 m is rotating with an angular velocity $\omega = 15.7$ rad/s and is transmitting a power $N = 11$ kW. Determine: (1) the rotational moment and (2) the peripheral force.

Solution. The rotational moment is found from (156):

$$M_{\text{rot}}=\frac{N}{\omega}=0.7\ \text{kN·m (kilonewton-metre)}.$$

Using formula (156), we find the peripheral force:

$$P=\frac{M_{\text{rot}}}{r}=3.5\ \text{kN}.$$

---

* If several forces are applied to a body, the total work and total power of these forces are found from similar formulas; in this case the rotational moment M_{rot} is the algebraic sum of the moments of all forces about the axis of rotation of the body.

Problem 104. The pulley A in Fig. 233 is rotated from the driving pulley B by means of a belt drive. The driving portion of the belt is under a tension of force $T = 2$ kN, the driven portion is under a tension of force $t = 1.2$ kN. The diameter of the pulley A is $2r = 600$ mm. Determine the work performed by the given forces over 10 rotations of pulley A and also power transmitted by the belt if the pulley has a rotational speed of 120 rpm.

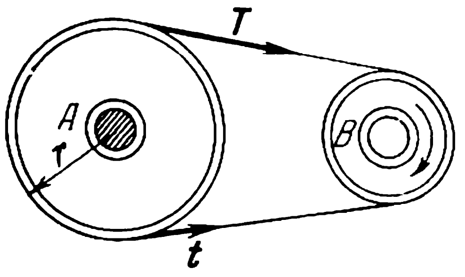


FIG. 233

Solution. The rotational moment applied to the pulley is

$$M_{\mathrm{rot}} = Tr - tr = (T - t)r$$
$$= (2 - 1.2) \times 0.3 = 0.24 \text{ kN}\cdot\text{m}.$$

The angle of rotation of the pulley $\varphi = 2\pi \times 10 = 62.8$ radians. By formula (154), the work of the forces applied to the pulley is

$$A = M_{\mathrm{rot}}\varphi = 0.24 \times 62.8 \approx 15 \text{ kN}\cdot\text{m}.$$

The angular velocity of the pulley corresponding to 120 rpm is

$$\omega = \frac{\pi n}{30} = \frac{3.14 \times 120}{30} = 12.56 \text{ rad/s}.$$

The power transmitted by the belt is found from (155) to be

$$N = M_{\mathrm{rot}}\omega = 0.24 \times 12.56 \approx 3.05 \text{ kW}.$$

CHAPTER XX

GENERAL THEOREMS ON THE DYNAMICS OF A POINT

Sec. 94. Momentum and impulse of a force

The momentum of a particle is the vector $m\mathbf{v}$ *which is equal to the product of the mass m of the particle by the vector* $\mathbf{v}$ *of its velocity.*

Since the mass of a particle is a positive scalar, the direction of the vector of the momentum of a particle always coincides with the direction of its velocity, whereas the magnitude of the momentum is equal to the product of the mass of the particle by the magnitude of the velocity of the particle. From the coincidence of the directions of the vectors $m\mathbf{v}$ of momentum and of the velocity $\mathbf{v}$ of the particle and from the relationship between their magnitudes it follows that the *projections of the momentum of a particle on coordinate axes are equal to the products of the mass of the particle into the appropriate projections of its velocity*:

$$(m\mathbf{v})_x = mv_x \quad \text{and} \quad (m\mathbf{v})_y = mv_y. \tag{157}$$

Momentum has the following dimensions:

$$[\text{momentum}] = [\text{mass}] \times [\text{velocity}] = \frac{[\text{force}] \times [\text{time}]^2}{[\text{length}]} \times \frac{[\text{length}]}{[\text{time}]} = [\text{force}] \times [\text{time}].$$

Thus, in the International System (SI) momentum is measured in kilogram-metres per second, that is, in kg·m/s.

Closely connected with the concept of momentum is that of the impulse of a force.

We consider only the case where a particle is acted upon by a force constant in magnitude and direction.

The impulse of a (constant in magnitude and direction) force over a certain time interval is a vector equal to the product of the vector $\mathbf{P}$ *of the force by the given time interval.*

Denoting the vector of the impulse of the force by **I**, we have

$$\mathbf{I} = \mathbf{P}t. \tag{158}$$

Since time is a scalar quantity, the direction of the vector **I** coincides with the direction of the force vector **P**.

Clearly, *the projections on the coordinate axes of the impulse of a constant force over a certain time interval are equal to the products of the appropriate projections of that force into the given time interval*:

$$I_x = Xt \quad \text{and} \quad I_y = Yt, \tag{159}$$

where X and Y are the projections of the force **P** on the corresponding coordinate axes.

The impulse of a force has the following dimensions:

$$[\text{impulse}] = [\text{force}] \times [\text{time}].$$

In the International System (SI), the unit of impulse of a force is the newton-second, that is, the impulse of a force equal to 1 N and acting during 1 second. Recalling the dimensions of a newton, we have

$$\mathrm{N \cdot s = kg \cdot \frac{m}{s^2} \cdot s = kg \cdot m/s}.$$

It is thus evident that the impulse of a force has the same dimensions as momentum.

Sec. 95. A theorem on the change of momentum of a particle

Suppose a particle of mass m is moving under the action of an applied force **P** relative to some fixed* system of coordinates along a trajectory AB (Fig. 234). Suppose the particle was located at M_0 at time $t = 0$ and had a velocity $\mathbf{v}_0$ at the beginning of the time interval, and reached position M_1 at time t_1 (end of the time interval) with a velocity $\mathbf{v}_1$.

---

* Here and elsewhere in dynamics, a fixed coordinate system is to be understood as a reference frame assumed to be an inertial frame.

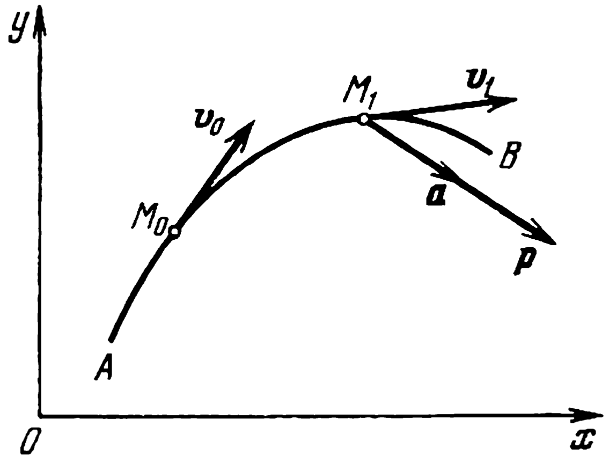


FIG. 234

The relationship between the acceleration vector **a** of the particle and the vector **P** of the force applied to it is given, as we know, by the basic equation of dynamics (128):

$$m\mathbf{a} = \mathbf{P}.$$

Projecting this vector equation on the x-axis, we get

$$ma_x = P_x = X.$$

But, as we know from kinematics [Sec. 60, formula (74)], the projection of acceleration on a fixed coordinate axis is equal to the first derivative, with respect to time, of the projection of the velocity of the particle on the same axis:

$$a_x = \frac{dv_x}{dt}.$$

Substituting this value into the preceding equation, we obtain

$$m\frac{dv_x}{dt} = X$$

or, bringing the constant factor m under the sign of the derivative in the left-hand member of the equation,

$$\frac{d}{dt}(mv_x) = X. \tag{160}$$

The derivative, with respect to time, of the projection on some axis of the momentum of a particle is equal to the projection on that axis of the force acting on the particle.

Multiplying both sides of (160) by dt, we get

$$d(mv_x) = X\ dt.$$

Integrating both sides of the equation within the appropriate limits and denoting the projection of the initial (at time $t = 0$) velocity of the particle on the x-axis by v_{0x}

and the projection of the terminal velocity of the particle (at time $t = t_1$) on the same axis by v_{1x}, we get

$$\int_{v_{0x}}^{v_{1x}} d(mv_x) = \int_0^{t_1} X\,dt.$$

Since the integral of $d\,(mv_x)$ is equal to mv_x and the projection X of the constant force **P** on the fixed axis is also a constant and may be taken outside the integral sign, then after substituting the limits of integration we obtain

$$mv_{1x} - mv_{0x} = Xt_1 = I_x. \tag{161}$$

In the same way we obtain a similar equation involving projections on the y-axis:

$$mv_{1y} - mv_{0y} = Yt_1 = I_y. \tag{162}$$

The equations (161) and (162) express the contents of a theorem on the change in the momentum of a particle (in terms of projections on the coordinate axes) that may be stated as follows: *the change in the projection of momentum of a particle on some axis is equal to the projection on that axis of the impulse of the force acting on the particle during the same time interval.**

In the case of rectilinear motion of a particle along the x-axis, the theorem is given by the single equation (161).

The theorem on the change in momentum of a particle, like the other basic theorems of dynamics, is a consequence of the second basic law of dynamics and the basic equation of dynamics that follows from it and is the result of a mathematical transformation of that equation.

---

* This theorem on the change in momentum of a particle holds true also for the case where the particle is acted upon by a variable force. Here the projections of the impulse of the force on the coordinate axes are equal to $I_x = \int_0^{t_1} X\,dt$ and $I_y = \int_0^{t_1} Y\,dt$. The projections X and Y of the force are variable quantities here and so cannot be taken outside the signs of the integrals. The values of the given integrals may be computed only when the projections of the force on the coordinate axes are known functions of time, that is, when we know the functions $X = f_1\,(t)$ and $Y = f_2\,(t)$.

The theorem on the change in momentum of a particle is useful for solving problems in which a relationship is established between the mass (or weight) of a particle, its velocity at the initial and terminal times of motion, the force and the time of action of the force; one of these quantities is sought and the others are known.

If a particle is acted upon simultaneously by several forces, then by the projections X, Y, Z of the force acting on the particle one must (by the law of independence of action of forces) understand the projection of the resultant of all forces applied to the particle; this projection, as we know, is equal to the algebraic sum of the projections of the component forces on the appropriate axis.

Any constrained particle, that is, a particle whose motion is restricted by certain conditions (constraints), may be (mentally) relieved of such constraints by substituting the constraint reactions for their action on the particle. Thus, when regarding the motion of a constrained particle we must include (among the forces acting on it) the reactions of all constraints imposed on the particle.

Problem 105. A body of mass $m = 20$ kg is in translational motion along a smooth horizontal plane with velocity $v_0 = 0.5$ m/s.

Determine the magnitude and direction of the velocity $\mathbf{v}_1$ of the body 3 seconds after the application to it of a constant force $P = 40$ N in the opposite direction to its initial velocity $\mathbf{v}_0$.

Solution. Since the weight of the body is balanced by the normal reaction of the horizontal plane, the resultant of all forces acting on the body during its motion over the smooth plane is the force **P**.

Let us take the direction of that force for the direction of the x-axis of the projections and apply to the given motion of the body the theorem on the change in momentum of a particle in the projection on the y-axis [equation (161)]:

$$mv_{1x} - mv_{0x} = I_x = Xt_1.$$

For the chosen direction of the projection axis, we have

$$v_{0x} = -v_0, \qquad X = P = \text{constant}.$$

Substituting these values into the equation, we obtain

$$mv_{1x} + mv_0 = Pt,$$

whence the projection of the terminal velocity $\mathbf{v}$ of the body is

$$v_{1x} = \frac{P}{m} \times t - v_0 = \frac{40}{20} \times 3 - 0.5 = 5.5 \text{ m/s}.$$

Since the projection of the velocity $\mathbf{v}_1$ is positive, the direction of this velocity coincides with the chosen direction of the x-axis, that is, with the direction of the force **P**. The magnitude of the desired velocity is

$$v_1 = |v_{1x}| = 5.5 \text{ m/s}.$$

Problem 106. A hammer weighing $G = 20$ kN falls from a height $h = 1$ m onto a workpiece being forged. Deformation of the workpiece occurs in a time $t_1 = 0.01$ s. Find the average force of pressure of the hammer on the workpiece.

Solution. The hammer is acted on by vertical forces: the force of gravity **G** and the reaction of the workpiece. We replace the variable quantity of the reaction of the workpiece (it changes in the course of the given time interval t_1) by a certain average value R.

We take the direction of motion of the hammer (vertically downwards) for the direction of the x-axis and apply the theorem on the change in momentum to the motion of the hammer over the time interval from the time of contact with the workpiece to the time when deformation is complete and the velocity of the hammer is zero:

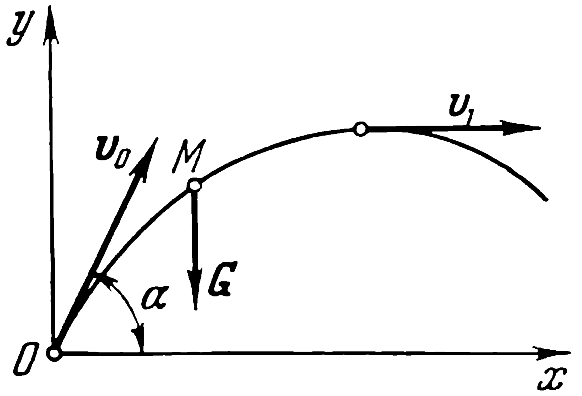


FIG. 235

$$mv_{1x} - mv_{0x} = I_x = Xt_1.$$

In this equation, the terminal velocity $\mathbf{v}_1$ of the hammer and its projection v_{1x} are zero. The initial velocity of the hammer, by Galileo's formula [formula (90)], $v_0 = \sqrt{2gh}$ and is directed vertically downwards. Hence, $v_{0x} = v_0 = \sqrt{2gh}$. The projection of the resultant of the forces acting on the hammer is

$$X = G - R = \text{constant}.$$

Substituting these values into the equation, we get

$$-\frac{G}{g}\sqrt{2gh} = (G - R)\,t_1.$$

From this, the reaction of the workpiece, which is equal in magnitude to the desired force of pressure of the hammer, is

$$R = G\left(1 + \frac{1}{t}\sqrt{\frac{2h}{g}}\right) = 20\left(1 + \frac{1}{0.01}\sqrt{\frac{2\times 1}{9.81}}\right) \approx 920 \text{ kN}.$$

Problem 107. A body* M is thrown at an angle α to the horizon (Fig. 235) with an initial velocity $\mathbf{v}_0$. Disregard air resistance and find the time t_1 of ascent of the body to the highest position.

---

* The body is in translational motion and so it may be regarded as a particle.

Solution. We disregard air resistance and so can consider that a constant (for the given locality) force is applied to the body: the force of gravity $G = mg$ directed vertically downwards.

Draw the coordinate axes as shown in Fig. 235 and apply the theorem on the change in momentum of a particle in the projection on the y-axis:

$$mv_{1y} - mv_{0y} = I_y = Yt_1.$$

For the chosen direction of the projection axis we have

$$v_{0y} = v_0 \sin \alpha, \qquad v_{1y} = 0$$

(since the velocity is horizontal at the highest position of the body);

$$Y = -G = -mg = \text{constant}.$$

Substituting these values into the equation, we get

$$-mv_0 \sin \alpha = -mgt_1,$$

whence we find the desired time of ascent of the body:

$$t_1 = \frac{v_0 \sin \alpha}{g}.$$

Sec. 96. Potential and kinetic energy

Energy is one of the basic physical concepts.

In mechanics, the energy of a body is a quantity that characterizes the capacity of the body to perform work under certain conditions. Two types of mechanical energy are distinguished: potential energy and kinetic energy.

Potential energy (or the energy of position) is energy that depends solely on the mutual positions of bodies or parts of one and the same body.

Potential energy of a body is measured by the work that it can perform upon displacement from the given position to some other position. For instance, a body weighing G and held at a height h above the earth's surface possesses (relative to that surface) a potential energy equal to the product Gh, that is, to the work it can perform in falling to the earth (if the obstacles to such motion of the body are removed).

It must be borne in mind that the concept of potential energy is a relative concept and is meaningful only when two positions of the body are indicated. It is obvious, for example, that the potential energy of a body lying on the ground at the edge of a well is zero with respect to the given site on the earth's surface. But that same body possesses a definite potential energy with respect to the bottom of the well, the energy being the greater, the deeper the well.

Also, a body in a state of elastic deformation possesses potential energy; for instance, a stretched, compressed or twisted spring, compressed air or compressed gas. The elastic forces that arise in deforming a body (that is, in changing the mutual arrangement of the particles of the body) perform a certain amount of work in restoring the shape of the body when the causes that hold the body in the deformed state are removed.

The kinetic energy of a body is the energy of its mechanical motion.

Kinetic energy is measured by the amount of work that a moving body can perform when brought to a halt. The kinetic energy of a particle (or a body in translational motion) depends only on the mass of the particle and its velocity at the start of deceleration. The greater the mass of the particle and the greater its velocity, the greater the kinetic energy the particle has.

To illustrate, take a shell moving at a high speed; when it encounters some obstacle like a wall or armour-plate, it loses speed in overcoming the resistance of such obstacles and in the process performs a tremendous amount of work.

As will be seen in the next section, *for the measure of kinetic energy of a particle, one should take half the product of the mass m of the particle into the square of the magnitude v of its velocity.*

Denoting the kinetic energy by T, we have

$$T=\frac{mv^2}{2}\,. \tag{163}$$

Let us determine the dimensions of this quantity:

$$\begin{aligned}[\text{kinetic energy}] &= [\text{mass}]\times[\text{velocity}]^2\\ &= [\text{mass}]\times\left[\frac{\text{length}}{\text{time}^2}\right]\times[\text{length}]=[\text{force}]\times[\text{length}].\end{aligned}$$

Thus, as was to be expected, kinetic energy has the same dimensions as the work of a force and is measured in joules.

Sec. 97. A theorem on the change in kinetic energy of a particle

Suppose a particle M of mass m is in motion under the action of an applied force **P** along some curvilinear trajectory (Fig. 236).

The motion of any particle obeys the basic law of dynamics and the relationship between the force **P** acting on the particle and the acceleration **a** of the particle generated by the force may be written down as the familiar equation (128):

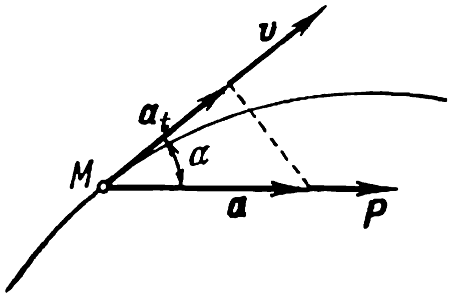


FIG. 236

$$m\mathbf{a} = \mathbf{P}.$$

According to that same law, the vectors of force **P** and acceleration **a** of the particle always coincide in direction. Denoting the angle between the direction of these vectors and the direction of the velocity vector **v** of the point by α and projecting both sides of this equation on the direction of velocity, we have

$$ma \cos \alpha = P \cos \alpha.$$

The product $a \cos \alpha$ is clearly the projection of the acceleration vector of the particle on the direction of the velocity of the particle (the direction of the tangent line to the trajectory of the particle), that is, nothing but the numerical value a_t of the tangential acceleration of the particle, which value is equal [as we know from kinematics, see Sec. 59, formula (70)] to the derivative of the numerical value of the velocity with respect to time:

$$a \cos \alpha = a_t = \frac{dv}{dt}.$$

Making the appropriate replacement in the left-hand member of the preceding equation and multiplying both sides of the equation by the infinitesimal displacement dS of particle M, we obtain

$$m \frac{dv}{dt} dS = P\, dS \cos \alpha.$$

Again transforming the expression in the left-hand member of this equation, we obtain

$$m\frac{dv}{dt}\,dS = m\frac{dS}{dt}\,dv = mv\,dv$$

since the derivative dS/dt is equal to the numerical value v of the velocity of the particle. We thus arrive at the equation

$$mv\,dv = P\,dS\cos\alpha.$$

Integrating both sides of this equation within the appropriate limits and denoting by v_0 the magnitude of the initial velocity of the particle, that is, the velocity of the particle corresponding to the time when the path S is zero, and by v_1 the magnitude of the terminal velocity of the particle, that is, the velocity corresponding to the time when the path length covered by the particle is equal to S, we have

$$\int_{v_0}^{v_1} mv\,dv = \int_0^S P\cos\alpha \times dS.$$

Taking the integral in the left-hand member and recalling [formula (141)] that the integral in the right-hand member of this equation is the work A of the force **P** applied to the given particle over its path S, we finally get

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A. \qquad (164)$$

According to the definition given in the preceding section, the expressions $mv_1^2/2$ and $mv_0^2/2$ constitute the kinetic energy of the particle at the terminal and initial times, respectively, of its motion. Hence, the equation (164) may be stated thus: *the change in the kinetic energy of a particle over a certain path is equal to the work of the force applied to it over that path* (the theorem on the change in kinetic energy of a particle).

If the motion of the particle took place under the action of several applied forces instead of one, then by the work A in (164) we must understand the work of the resultant of these forces; this work, as was demonstrated in Sec. 89, is equal to the algebraic sum of the works of all component forces.

If under the action of forces of resistance applied to the given particle, the particle stops, then the terminal velocity v is equal to zero and equation (164) takes the form

$$-\frac{mv_0^2}{2}=A.$$

The work of resistance forces is always negative, as we know, and so the right-hand side of the above equation is negative too. Thus, the expression $mv_0^2/2$ is the work which can be performed by a particle moving with velocity v_0 when decelerated to a halt and, hence, this quantity does indeed serve as a measure of the kinetic energy of the particle.

The theorem on the change in kinetic energy of a particle permits determining the work of applied forces when the particle moves from one position to another and in those cases (of a variable force and a curvilinear motion of the particle) when direct calculation of the work by means of formula (141) is difficult. The only thing that must be known is the mass of the particle and the magnitudes of its velocity in the initial and terminal positions.

If, on the contrary, we are able to find the work of the applied forces, then, knowing the mass of the particle and the magnitude of its velocity in one position, it is easy to find (using the given theorem) the magnitude of the velocity of the particle in the other position.

The theorem on the change in the kinetic energy of a particle yields the simplest method for solving problems in which a relationship is established between the force acting on the particle, the velocity of the particle, and the path traversed.

Problem 108. Referring to Fig. 237, a particle of mass m attached to a fixed point O by a string of length l, is moved through angle α from the equilibrium position OM and is released at point M_0 without an initial velocity. Find the velocity $\mathbf{v}_1$ of the particle at the time it passes through the equilibrium position.

Solution. By the theorem on the change in kinetic energy of a particle,

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=A.$$

The displacement of the point from M_0 to M is due to the action of the force of gravity. The work of this force is, by (142),

$$A = Gh = G \times AM = G(OM - OA) = G(l - l\cos\alpha)$$

$$= mgl(1 - \cos\alpha) = 2mgl\sin^2\frac{\alpha}{2}.$$

Since initial velocity of the particle is $v_0 = 0$ the equation of the kinetic energy takes the form

$$\frac{mv_1^2}{2} = 2mgl\sin^2\frac{\alpha}{2},$$

whence

$$v_1 = \sqrt{4gl\sin^2\frac{\alpha}{2}} = 2\sqrt{gl}\sin\frac{\alpha}{2}.$$

Problem 109. Referring to Fig. 238, an automobile is moving up an incline at 54 km/h (the angle of the incline $\alpha = 10°$). Regarding the resistance force P due to braking as equal to 0.3 the weight of the vehicle and disregarding all other resistances to its motion, find the distance S it covers before stopping and the time t_1 that elapses from onset braking to a complete halt.

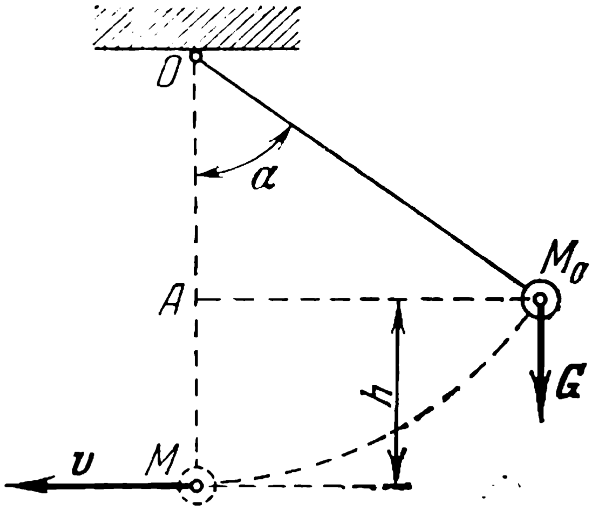


FIG. 237

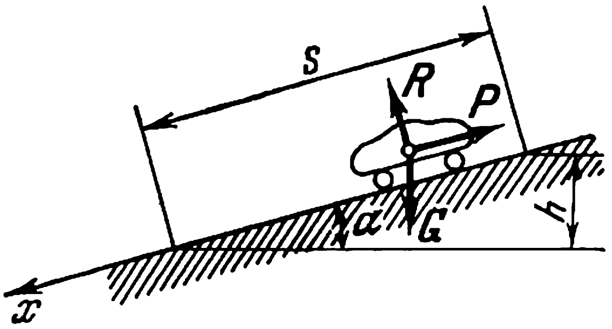


FIG. 238.

Solution. We regard the vehicle as a particle acted upon by the following forces: **G**, the force of gravity of the automobile, **R**, the normal reaction of the road, and **P**, the resistance due to braking.

To find the braking distance S, we apply the theorem on the change in kinetic energy of a particle:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A.$$

Given are the initial velocity of the vehicle $v_0 = 54$ km/h $= 15$ m/s and the terminal velocity $v_1 = 0$.

The work A of the resultant of the forces applied to the automobile is equal to the algebraic sum of the works of the component forces. The work of the force of gravity is $Gh = mgS \sin\alpha$; the work of the normal reaction **R** is zero since this force is perpendicular to the direction of motion of the vehicle, the work of the braking force **P** is

$$PS\cos\alpha = -0.3mgS$$

since the given force is in the opposite direction to the motion of the automobile and, hence, $\cos\alpha = -1$. Thus, the work of the resultant is

$$A = mgS\sin\alpha - 0.3mgS = mgS(\sin\alpha - 0.3).$$

For the given case, the equation of the kinetic energy takes the form

$$-\frac{mv_0^2}{2} = mgS(\sin\alpha - 0.3)$$

whence

$$S = -\frac{v_0^2}{2g(\sin\alpha - 0.3)} = -\frac{15^2}{2\times 9.81(0.174 - 0.3)} \approx 91 \text{ m}.$$

To find the braking time t_1, we can take advantage of the theorem on the change in momentum of a particle. Taking the direction of motion of the automobile as the direction of the x-axis, we can write equation (161):

$$mv_{1x} - mv_{0x} = I_x.$$

In this case, $v_{1x} = 0$, $v_{0x} = v_0 = 15$ m/s. Since the forces acting on the automobile are constant, the projection on the x-axis of the impulse of these forces is

$$I_x = Xt_1 = (G\sin\alpha - P)t_1 = (mg\sin\alpha - 0.3mg)t_1$$
$$= mgt_1(\sin\alpha - 0.3).$$

Thus, the equation of the momentum of the particle here takes the form

$$-mv_0 = mgt_1(\sin\alpha - 0.3),$$

whence

$$t_1 = -\frac{v_0}{g(\sin\alpha - 0.3)} = -\frac{15}{9.81(0.174 - 0.3)} \approx 12 \text{ s}.$$

Problem 110. A pile hammer of mass $m = 100$ kg is raised to a height of $h = 1$ m; on the last drop, the pile is driven into the ground to a depth of $S = 1$ cm. What is the greatest pressure (in kPa) that the pile can stand up to without sinking if we regard the resistance of the ground to the motion of the pile as constant and if we ignore the mass of the pile? The pile has a cross-section of $F = 0.15$ m^2.

Solution. We apply to the motion of the pile hammer the equation of kinetic energy of a particle:

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=A.$$

The pile hammer is acted upon by the following forces: the force of gravity **G** (throughout its displacement, which is equal to $h+S$) and the reaction **R** of the pile, which is equal to the resistance force of the ground. This latter force obviously acts on the pile hammer only over the displacement of the pile, which is equal to S.

The force of gravity

$$G=mg=100\times 9.81=981\ \text{N}.$$

The work of gravity is equal to $G(h+S)$. The work of the reaction of the pile is

$$RS\cos\alpha=-RS$$

since the direction of the reaction R is opposite to the motion of the pile hammer and $\cos\alpha=-1$.

Since the velocity of the pile hammer both at the beginning and at the end of motion is zero, the kinetic energy of the hammer does not change over the displacement $h+S$. Hence, during this displacement the algebraic sum of the works of all forces applied to it must be zero.

We thus get

$$A=G(h+S)-RS=0,$$

whence we find the reaction of the pile:

$$R=\frac{G(h+S)}{S}=\frac{981\times 101}{1}=99\,081\ \text{N}\approx 99.1\ \text{kN}.$$

The pressure p that the pile can withstand without sinking is equal to the specific resistance of the ground. Hence,

$$p=\frac{R}{F}=\frac{99.1}{0.15}=660\ \text{kPa}.$$

Problem 111. A beam (Fig. 239) rests on two supports and is loaded at the midpoint with a load M of weight G; when vibrations cease, the sag is $f=1$ mm. Find the largest sag h of the beam in two cases: (1) when the load M falls on the midpoint of the unbent beam from a height of $H=20$ cm without initial velocity, and (2) when the load M is placed on the unbent beam and released without initial velocity.

Solution. The load M is acted upon by the following forces: (1) the force of gravity **G** throughout the displacement of the load from position

M_0 to M' [which corresponds to the position of greatest sag (deflection) of the beam] that is, over the displacement $H+h$; (2) the elastic reaction **R** of the beam that obviously acts on the load only over the displacement h, that is, over the displacement corresponding to the elastic deformation of the beam.

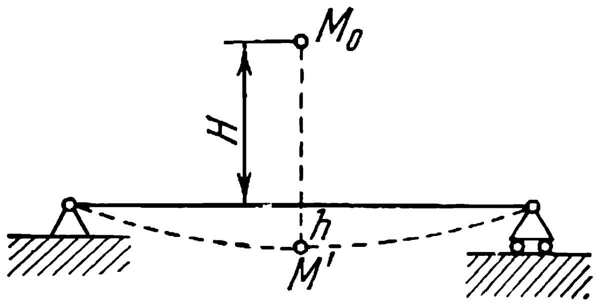


FIG. 239

The work of gravity is equal to $G(H+h)$, the work of the elastic force of the beam is, by formula (144), equal to $-ch^2/2$. The stiffness coefficient c of the beam is found from the conditions of the problem. In the position of equilibrium of the load lying on the elastic beam (often vibrations of the beam have ceased), the elastic force of the beam is equal to the weight of the load. From formula (143) we find $c = P/x$. In the given case, $P = G$ and $x = f$, whence $c = G/f$.

Thus the work of the elastic force of the beam at maximum deflection (sag) equal to h is

$$-\frac{Gh^2}{2f}.$$

Since the velocity of the load both in its initial position M_0 and in the terminal position M' is zero, there is no change in the kinetic energy of the load over its displacement $H+h$. Hence, the sum of the works of all forces applied to the load must be zero over this displacement:

$$A = G(H+h) - \frac{Gh^2}{2f} = 0$$

or

$$h^2 - 2fh - 2fH = 0.$$

Solving this quadratic equation for the unknown h (maximum deflection), we find

$$h = f + \sqrt{f^2 + 2fH} = 1 + \sqrt{1^2 + 2 \times 1 \times 200} \approx 21 \text{ mm}.$$

The plus sign is chosen in this solution because otherwise the deflection h would be negative, which is meaningless.

2. When the load is placed on the unbent beam and released without initial velocity, the height of fall of the load is $H = 0$. Substituting this value into the solution of the quadratic equation, we get

$$h = f + \sqrt{f^2 + 2f \times 0} = 2f = 2 \times 1 = 2 \text{ mm},$$

CHAPTER XXI

ELEMENTS OF THE DYNAMICS OF A SYSTEM

Sec. 98. External and internal forces of a system

As we already know, a system of particles, or simply a system, is a collection of material points (particles) connected by certain conditions. Thus, the motion of each particle of the system depends on the motions of all other particles.

Every rigid body may be regarded as a system of particles. A collection of bodies associated in some fashion is also a system. For instance, in studying the motion of the crank gear shown in Fig. 240 of the engine of a motor car, we may (depending on the nature of the problem at hand) take for the system the collection of all the interrelated units of the mechanism or any part of the mechanism, say a separate unit.

All forces acting on the particles of the given system may be divided into two types: internal forces and external forces.

The internal forces are forces of interaction between particles or bodies of the system. We will denote such an internal force by $\mathbf{P}^i$. *The external forces are forces outside the given system that act on the particles of the system.* We will denote any such external force by $\mathbf{P}^e$.

The division of forces into internal and external forces is an artificial one and depends on what is included in the system at hand. Thus, taking the above example, if we regard the piston B as a separate system, then the forces of the other units of the mechanism (the cylinder of the engine and the connecting rod AB) must be regarded as external forces. In this case, the internal forces are only forces of interaction between the particles of the piston. If we regard the entire crank gear of the engine as a system, then the forces of interaction between the separate units must be regarded as internal forces. The pressure of gases on a piston of the engine is an external force with respect to the crank gear. Now, if we regard as one system the motor car as a whole together with the engine, then the pressure of the gases on a piston of the engine will be an internal force. In such a system, the external forces will include: the weight

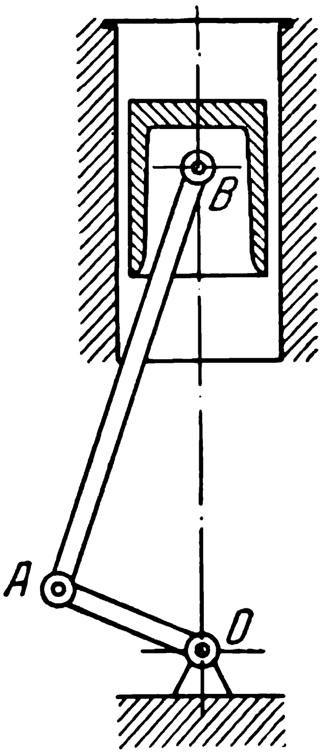


FIG. 240

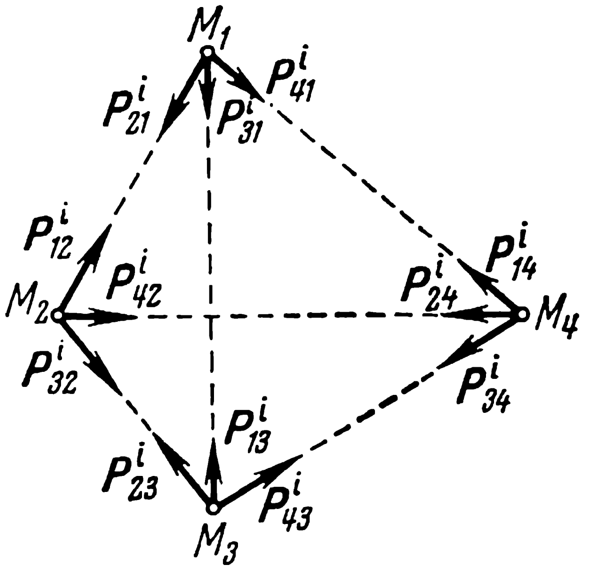


FIG. 241

of the car, the normal reaction of the road, the friction between the wheels and the surface of the road, and the air resistance.

By the law of the equality of action and reaction, the forces of interaction between any two particles are always equal in magnitude and opposite in direction.

Let us imagine the particles M_1, M_2, M_3, M_4 (Fig. 241) of some system. The force $\mathbf{P}^i_{12}$ with which particle M_1 of the system acts on particle M_2 of the same system will clearly be equal in magnitude and opposite in direction to the force $\mathbf{P}^i_{21}$ with which particle M_2 acts on particle M_1. The force $\mathbf{P}^i_{13}$ with which particle M_1 acts on particle M_3 will be equal in magnitude and opposite in direction to the force $\mathbf{P}^i_{31}$ with which particle M_3 acts on particle M_1. Similarly, $\mathbf{P}^i_{14} = -\mathbf{P}^i_{41}$, $\mathbf{P}^i_{24} = -\mathbf{P}^i_{42}$ and so forth. By combining in pairs the forces of interaction between all particles of the given system (internal forces of the system), we arrive at the conclusion that the *geometric sum of all internal forces of any system is equal to zero*:

$$\sum \mathbf{P}_k^i = 0. \qquad (165)$$

From this equation it follows that, also, the *algebraic sum of the projections on any axis of all internal forces of a system is equal to zero*:

$$\sum X_k^i = 0, \quad \sum Y_k^i = 0, \quad \sum Z_k^i = 0. \qquad (166)$$

These conclusions [formulas (165) and (166)] sometimes make it possible to greatly simplify the investigation of questions referring to systems of particles because they permit dispensing with the internal forces of a system in certain cases.

Sec. *99. The law of conservation of momentum of a system

The momentum of a system, **Q**, *is the geometric sum of the vectors of the momenta of all particles of the system*:

$$\mathbf{Q} = \sum m_k \mathbf{v}_k \qquad (167)$$

where m_k is the mass of a particle of the system and $\mathbf{v}_k$ is the velocity vector of that particle.

Since the projection of the geometric sum of the vectors on any axis is equal to the algebraic sum of the projections on that axis of the component vectors, it follows that

$$Q_x = \sum m_k v_{kx}. \qquad (168)$$

The projection of the momentum of a system on some axis is equal to the algebraic sum of the projections on that axis of the momenta of all particles of the system.

Let us take advantage of the formula (160) given in Sec. 95 for the motion of a single particle:

$$\frac{d}{dt} mv_x = X,$$

where X is the projection of the resultant of all forces acting on the particle.

We divide the forces acting on the particles of the system into external and internal forces. Denote by X_k^e the projection, on the x-axis, of the resultant of external forces applied to some kth particle of the system with mass m_k, by X_k^i the projection, on that same axis, of the resultant of the internal forces applied to that particle and by v_{kx} the projection, on the x-axis, of the velocity of the given particle.

On the basis of formula (160), we can write, for the chosen particle of the system,

$$\frac{d}{dt} m_k v_{kx} = X_k^e + X_k^i.$$

Writing similar equalities for all particles of the given system and combining them termwise, we obtain

$$\sum \frac{d}{dt} m_k v_{kx} = \sum X_k^e + \sum X_k^i.$$

But, as was established in the preceding section, the sum of the projections, on any axis, of all internal forces of a system is $\sum X_k^i = 0$. The sum of the derivatives is, as we know, equal to the derivative of the sum of the terms. We can therefore write

$$\frac{d}{dt} \sum m_k v_{kx} = \sum X_k^e. \tag{169}$$

The derivative of the projection of the momentum of a system on some fixed axis is equal to the sum of the projections, on that axis, of all external forces acting on the system. Since the derivative of a constant is equal to zero, it follows from (169) that

$$\sum m_k v_{kx} = \text{constant} \quad \text{if} \quad \sum X_k^e = 0. \tag{170}$$

*If in the course of some time the sum of the projections, on some fixed axis, of all external forces of a system remains zero**,

---

* The geometric sum of the external forces acting on the system may not be zero. For example, if the system is acted upon by vertical forces alone (even though they might all be in the same direction), the sum of their projections on the horizontal axis is equal to zero.

then the projection, on that axis, of the momentum of the system during that time remains constant.

It is clear that if the geometric sum of all external forces of a system is zero, $\sum \mathbf{P}_k^e = 0$, then so also will the sum of the projections of these forces on any axis be zero. Hence, in this case the projection of the momentum of the system will be constant too (and on any axis also). This latter circumstance can only occur when the momentum itself of the system is constant, that is, when the geometric sum of the vectors of the momenta of all particles of the system is constant in both magnitude and direction.

From this proposition, which expresses the very important *law of conservation of momentum of a system*, it follows that the *internal forces cannot alter the total momentum of a system.*

For instance, if we regard a gun and a shell as a single system, then the pressure of powder gases when the gun is fired is an internal force. The pressure of the gases imparts a certain momentum to the shell and simultaneously imparts to the gun the same momentum in the opposite direction. This is called recoil of the gun (that is, if the gun is not fixed in position and the recoil is not countered by any force).

The phenomenon of recoil explains the acceleration of a rocket too. The fuel exhaust (combustion product of the fuel) is ejected backwards from the rocket with a high velocity. Since the forces of pressure of the gases are, for the given system, also internal forces, they cannot alter the total momentum of the system, and so the rocket moves in the opposite direction, that is, forwards.

Problem 112. A gun of mass $m_1 = 125\,000$ kg located on a smooth horizontal platform fires a shell of mass $m_2 = 350$ kg in the horizontal direction with a velocity $v_2 = 550$ m/s. Find the velocity of the gun after the shot (the rate of free recoil).

Solution. Since the pressure of the powder gases in the bore of the gun is an internal force, the only external forces acting on the system (gun and shell) are the forces of gravity of the gun and shell and the normal reaction of the platform. These external forces are vertical forces and so the sum $\sum X_k^e$ of their projections on the horizontal x-axis is zero. In this case, by equation (170),

$$\sum m_k v_{kx} = \sum m_k v_{0kx} = \text{constant},$$

Prior to firing, the system was stationary and so $\sum m_k v_{0kx} = 0$. From this it follows that also after the shot was fired the projection of the momentum of the system on the horizontal x-axis is

$$\sum m_k v_{kx} = m_1 v_{1x} + m_2 v_2 = 0.$$

From this equation we find the projection of the desired velocity of free recoil of the gun:

$$v_{1x} = -\frac{m_2}{m_1} v_2 = -\frac{350}{125\,000} \times 550 = -1.54 \text{ m/s}.$$

The minus sign shows that the velocity $\mathbf{v}_1$ of recoil of the gun is in the opposite direction to the velocity of the shell, $\mathbf{v}_2$.

Sec. *100. The centre of mass of a system. A theorem on the motion of the centre of mass of a system

The centre of mass of a system is the geometric point C whose position is given by the following coordinates:

$$x_C = \frac{\sum m_k x_k}{M}, \quad y_C = \frac{\sum m_k y_k}{M}, \quad z_C = \frac{\sum m_k z_k}{M}. \tag{171}$$

In these formulas, m_k is the mass of an arbitrary particle of the system, x_k, y_k, z_k are the coordinates of the particle, $\sum m_k x_k$, $\sum m_k y_k$, $\sum m_k z_k$ are the algebraic sums made up of the products of the mass of each particle of the given system into the corresponding coordinates of the particle, and M is the mass of the entire system.

Each of the formulas (171) may be cast in a different form if we multiply the numerator and denominator of its right member by the acceleration g of a freely falling body. We then get

$$x_C = \frac{g \sum m_k x_k}{Mg} = \frac{\sum m_k g x_k}{Mg}.$$

But the product $m_k g$ is equal to the weight G_k of one particle, and the product Mg is equal to the weight G of th whole system. Substituting these values into the precedin formula and writing similar expressions for the other two coordinates of the centre of mass, we obtain

$$x_C = \frac{\sum G_k x_k}{G}, \quad y_C = \frac{\sum G_k y_k}{G}, \quad z_C = \frac{\sum G_k z_k}{G}.$$

As we know from statics [Sec. 46, formulas (48)], these formulas define the coordinates of the centre of gravity of a body. However, the concepts of centre of gravity and centre of mass are not identical.

The centre of gravity of a system is a point through which the resultant of the forces of gravity of all particles of the given system passes. The concept of centre of gravity is thus applicable only to systems that are located in the field of terrestrial gravitation and is meaningless relative to a system of bodies like the solar system. Now, the position of the centre of mass as defined at each given instant of time by the formulas (171) depends only on the masses of the particles of the system and on the positions of these particles at a given instant of time. The concept of centre of mass retains its meaning for any mechanical system, irrespective of the forces acting on it and, consequently, is a broader notion than that of the centre of gravity.

Let us consider the motion of a material system.

From formula (171), $x_C = \frac{\sum m_k x_k}{M}$, for the coordinate of the centre of mass of the system we have

$$Mx_C = \sum m_k x_k,$$

where M is the mass of the whole system, x_C is the time-varying coordinate of the centre of mass of the moving system, m_k is the mass of some kth particle of the system, and x_k is also a time-varying coordinate of that particle of the system.

Taking the derivatives of both sides of the given equation with respect to time, we obtain

$$M\frac{dx_C}{dt} = \frac{d}{dt}\sum m_k x_k = \sum m_k \frac{dx_k}{dt}.$$

But the derivative dx_C/dt is the projection v_{Cx} on the x-axis of the velocity $\mathbf{v}_C$ of the centre of mass, and the derivative dx_k/dt is the projection v_{kx}, on that axis, of the velocity of the kth particle of the system. Substituting these values into the preceding equation, we have

$$\sum m_k v_{kx} = M v_{Cx}. \tag{172}$$

The projection of the momentum of a system on some axis is equal to the projection, on that axis, of the momentum of the centre of mass of the system if it is assumed that the entire mass of the system is concentrated in the centre of mass.

Again, let us take the derivatives of both sides of (172). Taking into account that the derivative dv_{Cx}/dt of the projection of the velocity of the particle is equal to the projection a_{Cx} on that axis of the acceleration of that particle, we obtain

$$\frac{d}{dt}\sum m_k v_{kx} = M\frac{dv_{Cx}}{dt} = Ma_{Cx}. \tag{I}$$

On the other hand, by (169), we have

$$\frac{d}{dt}\sum m_k v_{kx} = \sum X_k^{\mathrm{e}}. \tag{II}$$

Since the left-hand members of (I) and (II) are equal, it is obvious that the right-hand members are equal too. Consequently,

$$Ma_{Cx} = \sum X_k^{\mathrm{e}}.$$

The expressions for the other two coordinates y_C and z_C of the centre of mass may be transformed in the very same way. We thus obtain the following three equations:

$$\left.\begin{aligned} Ma_{Cx} &= \sum X_k^{\mathrm{e}}, \\ Ma_{Cy} &= \sum Y_k^{\mathrm{e}}, \\ Ma_{Cz} &= \sum Z_k^{\mathrm{e}}. \end{aligned}\right\} \tag{173}$$

The right-hand members of these equations are the projections, on coordinate axes, of the vector $\sum \mathbf{P}_k^{\mathrm{e}}$ of the geometric sum of all external forces acting on the system, while the left-hand members are the projections, on the same axes, of the vector $M\mathbf{a}_C$, that is, the product of the mass of the system into the acceleration vector $\mathbf{a}_C$ of its centre of mass. The fact that the projections of these vectors on three coordinate axes are equal means the vectors themselves are equal. Hence, the three equations (173) express in coordinate form a single vector equation:

$$M\mathbf{a}_C = \sum \mathbf{P}_k^e. \tag{174}$$

Comparing this equation with the basic equation (122) of dynamics for a single particle, we can readily conclude that *equation* (174), *the equation of motion of the centre of mass, is an equation of motion of a particle of mass M to which are applied all external forces acting on the system.*

We can thus state the theorem on the motion of the centre of mass of a system as follows:

The centre of mass of a system moves like a particle whose mass is equal to the mass of the entire system and to which are applied all external forces acting on the given system (these forces are transferred to the centre of mass without any change in their magnitudes or directions).

Let us consider a number of important corollaries that follow from this theorem.

1. When a rigid body is in translational motion, its centre of gravity, which coincides with its centre of mass, moves in the same way that all other particles of the body do. By determining the motion of the centre of mass of such a body on the basis of the theorem of motion of that particle, we ipso facto determine the motion of the whole body as a consequence. Thus, as we have already done in working problems, *when investigating the translational motion of a body, the body may be regarded as a particle: we concentrate the entire mass of the body at its centre of gravity and transfer to the particle all external forces acting on the body.*

If the motion of the body is not translational, the compound motion may be decomposed into translational motion together with the centre of gravity and into rotational motion about the centre of gravity. The translational portion of that compound motion of a body is likewise fully determined by the theorem on the motion of the centre of mass of a body, that is, by equation (174). From this it follows that even in the case of nontranslational motion, a body of finite dimensions may be regarded as a particle; however, this can only be done if the rotational part of the motion is ignored. For example, in astronomy this is exactly what is done when studying the translational part of planetary motion.

2. The internal forces of a system, that is, the forces of interaction between the particles of the given system, do not enter into the equation (174) of the motion of the centre of mass. From this it follows that the *internal forces of a system do not affect the motion of its centre of mass*, just as they do not affect the total momentum of the system.

3. If $\sum \mathbf{P}_k^e = 0$, then, as follows from equation (174), $\mathbf{a}_C = 0$ and therefore $v_C =$ constant, that is, here the velocity of the centre of mass is a constant both in magnitude and in direction. In particular, if the initial velocity v_C of the centre of mass is equal to zero, then the centre of mass will remain at rest.

Consequently, *if during a certain time the geometric sum of all external forces acting on a system is equal to zero, the centre of mass of the given system will all the time remain at rest or in uniform motion in a straight line.*

4. If $\sum X_k^e = 0$, then, as follows from equation (173), $a_{Cx} = 0$ and therefore $v_{Cx} = dx_C/dt =$ constant.

Hence, *if during a certain time the sum of the projections, on some axis, of all the external forces acting on a system is equal to zero, then the projection of the velocity of the centre of mass on that axis will all the time remain constant.* In particular, if at an initial time the projection, on a given axis, of the velocity of the centre of mass $v_{Cx} = dx_C/dt = 0$, then the corresponding coordinate of the centre of mass will remain constant ($x_C =$ constant).

Let us now examine some examples that illustrate these propositions.

1. If air resistance is ignored, the only external force acting on a shell in flight is the force of gravity of the shell. Therefore the centre of gravity of the shell will move in the same way that any particle (in a vacuum) thrown at an angle to the horizon does, that is, in a parabola. When a shell bursts in flight, the fragments fly off in different directions, but their centre of mass continues the earlier motion until at least one of the fragments reaches the ground; as a result, to the external forces acting on the system there will

be adjoined the reaction of the ground, and this will alter the motion of the centre of mass. The forces that arise when the shell bursts are internal forces, and so they do not alter the motion of the centre of mass of the shell.

2. A person standing on an absolutely smooth horizontal surface cannot by himself move in the horizontal direction. Relative to this system (the person), the external forces are the weight of the person and the normal reaction of the smooth horizontal plane. Both of these forces are vertical forces, and therefore the sum of their projections on any horizontal axis is zero. Hence, the projection of the acceleration of the centre of gravity of the person on this axis is also zero. If the person was standing still at the beginning, then, due to the absence of any horizontal external forces, the horizontal velocity of its centre of gravity will remain zero. There is no way, by means of internal efforts, to displace the centre of gravity of the system. As soon as the person puts one foot forward, the other foot will immediately move backward and, consequently, the overall centre of gravity will remain in place. (To some extent, this is approximated in motion over smooth ice.) In actual conditions, locomotion of a person over a horizontal surface is possible only via the external horizontal force that is due to friction between the plane and the soles of one's shoes. When a person puts one foot forward, the other tends to go back, but this is prevented by the friction that develops between one's foot and the plane (surface); that is why the foot remains in place (or almost in place).

A similar explanation can be given for the horizontal motion of a locomotive, motor car and so forth. The pressure of gas on a piston of the engine of the locomotive is, with respect to the locomotive, an internal force and therefore cannot impart motion to the centre of gravity of the locomotive*. A train moves only because of the friction between the rails and the driving wheels of the locomotive. These wheels (Fig. 242) have irregularities and rough spots that press on the rails and experience a reverse pressure from the rails. This pressure, that is, the force of friction applied by the rail to the driving wheel in the direction of motion of the locomotive, constitutes the necessary horizontal external force that causes the centre of gravity of the locomotive to move.

---

* If a locomotive is hoisted on chains and the engine is started up, the body of the locomotive will only receive small horizontal deviations in the direction opposite to the motion of the piston, while its centre of gravity will remain stationary.

Above, we mentioned that the internal forces of a system cannot of themselves alter the motion of the centre of mass and the total momentum of the system. However, from the examples we have just considered it is evident that by generating motion of separate parts of a system and their action on external bodies and the material medium surrounding the system the internal forces are capable of generating external forces (relative to the given system) that can change the motion of the centre of mass of the system and its total momentum.

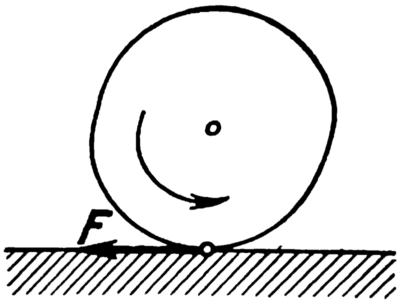


FIG. 242

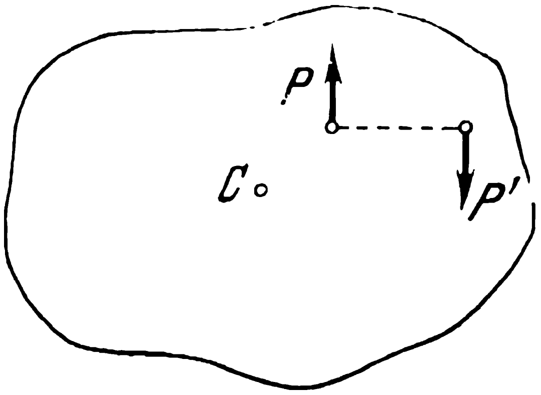


FIG. 243

3. Let us apply a force couple (**P**, **P**′) to a fixed rigid body (Fig. 243) that is capable of freely moving in space. Since the geometric sum of the forces of the couple is zero, it cannot impart acceleration to the centre of mass of the body and the body must remain stationary. This explains why a force couple always imparts rotation to a free body about the axis passing through the centre of mass of the body.

Problem 113. Referring to Fig. 244, a man is standing at one end, B, of a boat, which is at rest. The man then moves to the other end, point A. Disregarding the resistance of the water, determine the distance that the boat will move in the process if the boat weighs $G_1 = 1.5$ kN, the man weighs $G_2 = 0.5$ kN, and the length of the boat is $AB = l = 4$ m.

Solution. The boat together with the man is acted upon only by the vertical external forces $\mathbf{G}_1$ and $\mathbf{G}_2$ and also by the vertical buoyant (expulsive) force of the water. The sum of their projections on the horizontal x-axis is zero; consequently, on the basis of the centre-of-mass theorem, the horizontal velocity of the centre of mass of the given system (boat and man) must be constant. Since the system was stationary to begin with, the centre of mass of the system must remain stationary when the man moves in the boat.

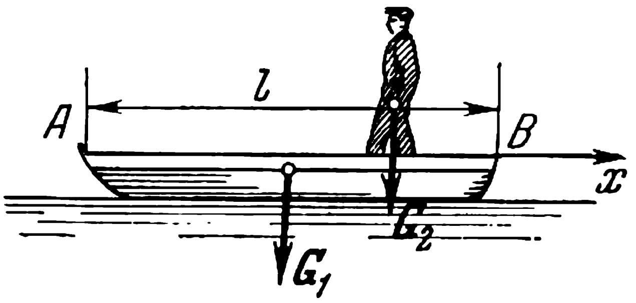


FIG. 244

For the fixed coordinate origin, take the original position of point A. The abscissa of the centre of mass of the system in the initial position of the system is

$$x'_C = \frac{G_1 \frac{l}{2} + G_2 l}{G_1 + G_2}.$$

Suppose, as the man moves from B to A, the boat is displaced rightwards a distance S. Then the abscissa of the centre of gravity of the system in its new position (with the same coordinate origin) is

$$x''_C = \frac{G_1\left(S + \frac{l}{2}\right) + G_2 S}{G_1 + G_2}$$

Since the centre of gravity of the system remains stationary, it follows that $x'_C = x''_C$, whence we find that

$$\frac{G_1 \frac{l}{2} + G_2 l}{G_1 + G_2} = \frac{G_1\left(S + \frac{l}{2}\right) + G_2 S}{G_1 + G_2}.$$

Solving this equation for S, we obtain

$$S = \frac{G_2 l}{G_1 + G_2} = \frac{0.5 \times 4}{1.5 + 0.5} = 1 \text{ m}.$$

Sec. 101. The basic equation of dynamics for rotational motion of a rigid body

Referring to Fig. 245, suppose a rigid body capable of rotating on a fixed axis z is under the action of forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$ applied to the system. As we know from statics (Sec. 43), the condition for equilibrium of a body having a fixed axis is that the algebraic sum of the moments of all action forces applied to the body, $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$, about the axis z of rotation of the body be zero.

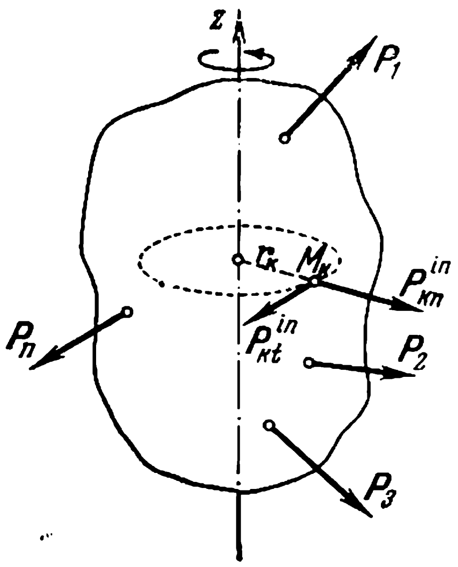


FIG. 245

However, if that sum of moments is not equal to zero, then the body will rotate about the z-axis with a certain angular acceleration $\varepsilon = d\omega/dt$, the sign of which obviously coincides with the sign of the principal moment of the forces applied to the body.

To determine the angular acceleration of the body, we take advantage of the method of dynamic force analysis. By adjoining (mentally) to the action forces applied to the body the forces of inertia of all its particles, we can apply the equilibrium condition to the resulting system of forces. In this case of the rotation of a body about a fixed axis, the equilibrium condition takes the following form: the sum of the moments of the action forces applied to the body and the forces of inertia of all its particles about the axis of rotation of the body must be equal to zero.

Partition the body into elementary particles. Denote the mass of one such particle M_k (Fig. 245) by m_k and its distance from the axis by r_k. Decompose the force of inertia of each such particle into components: a centrifugal component equal in magnitude to $P_{kn}^{\text{in}} = m_k r_k \omega^2$ and directed along a radius from the centre, and a tangential component equal in magnitude to $P_{kt}^{\text{in}} = m_k r_k \varepsilon$ and directed along the perpendicular to the radius in the direction opposite that of tangential acceleration. The directions of the forces $\mathbf{P}_{kn}^{\text{in}}$

and $\mathbf{P}_{kt}^{\text{in}}$ are shown in Fig. 245. We assume here that the principal moment of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$ applied to the body is positive (that is, it rotates the body counterclockwise when one looks from the positive end of the z-axis) and, consequently, the angular acceleration ε is positive as well. If the angular acceleration is negative, the direction of the tangential force of inertia is reversed.

All the centrifugal forces of inertia intersect the z-axis of rotation of the body, and, hence, their moments about that axis are zero. Relative to the axis of rotation of the body, the tangential force of inertia of particle M_k yields the moment*

$$m_z(\mathbf{P}_{kt}^{\text{in}}) = -P_{kt}^{\text{in}} r = -m_k r_k^2 \varepsilon.$$

Setting up the sum of the moments of the tangential forces of inertia for all elementary particles of the body, we will have

$$\sum m_z(\mathbf{P}_{kt}^{\text{in}}) = -\sum m_k r_k^2 \varepsilon = -\varepsilon \sum m_k r_k^2,$$

where the angular acceleration ε of the body is taken outside the sign of the sum (as a common factor).

We denote $\sum m_k r_k^2$ by J and call it the moment of the inertia of the body about its axis of rotation.

The moment J of inertia of the body about some axis is the sum made up of the products of the mass m_k of each particle of the body into the square of the distance r_k of that particle from the given axis.

Thus, *the sum of the moments of the forces of inertia of all particles of a body rotating about a fixed axis is equal to*

$$\sum m_z(\mathbf{P}_k^{\text{in}}) = -J\varepsilon. \qquad (175)$$

Adding this sum to the sum of the moments of all action forces applied to the body, we obtain (in accordance with the condition established above) the following equation:

---

* In the case of a negative principal moment of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, . . ., $\mathbf{P}_n$, the angular acceleration ε of the body will be negative and the moment $m_z(\mathbf{P}_{kt}^{\text{in}})$ will be positive. Thus the sign of the moment of the tangential force of inertia will always be opposite the sign of the moment of the forces applied to the body.

$$\sum m_z(\mathbf{P}_k)+\sum m_z(\mathbf{P}_k^{\text{in}})=\sum m_z(\mathbf{P}_k)-J\varepsilon=0.$$

For the sake of brevity we will call the algebraic sum of the moments of all action forces applied to the body about the axis of rotation (z-axis) of the body the *rotational moment (torque)* and we will denote it by M_{rot}:

$$M_{\text{rot}}=\sum m_z(\mathbf{P}_k). \qquad (176)$$

We thus arrive at the following equation:

$$M_{\text{rot}}=J\varepsilon. \qquad (177)$$

The rotational moment M_{rot} *of a body about a fixed axis is equal to the moment of inertia* J *of the body about that axis multiplied by the angular acceleration* ε *of the body. Equation* (177) *is called the basic equation of dynamics for the rotational motion of a rigid body about a fixed axis.*

Since the moment of inertia J of the given body about the given axis is a constant quantity, then, as follows from (177), *if the rotational moment* M_{rot} *is constant, the angular acceleration* ε *is constant, that is, the body exhibits uniformly variable rotation.*

From the same equation it follows that *if the rotational moment* M_{rot} *of the body is zero, then the angular acceleration of the body is also zero, that is, the body either remains at rest* (if it was already at rest) *or rotates with a constant angular velocity.*

For instance, if the moment of the moving forces applied to the shaft of a machine is equal, in magnitude,* to the moment of the forces of resistance, then the shaft will rotate uniformly. If the moment of the moving forces exceeds the moment of the forces of resistance, the shaft will be in accelerated rotation. If the moment of moving forces is less than the moment of the forces of resistance, the shaft will be in decelerated rotation.

---

* The sign of the moment of the forces of resistance is always opposite to the sign of the moment of the moving forces.

It will be readily seen that, in form, the basic equation of dynamics for rotational motion of a body, equation (177), resembles the basic equation (127) of dynamics for a particle (or, what is the same, for translational motion of a body):

$$P = ma.$$

In equation (177), in place of the magnitude P of the force we have the moment M_{rot} (rotational moment) of the force, in place of the mass m, the moment of inertia, J, and in place of the magnitude a of the linear acceleration, the angular acceleration ε.

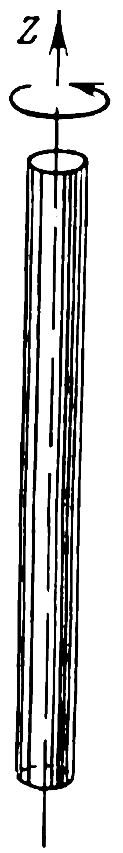


FIG. 246

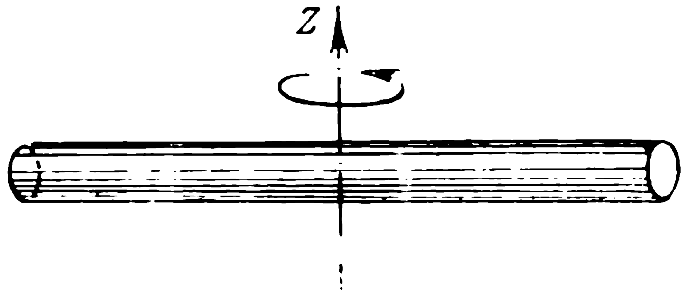


FIG. 247

It is evident from a comparison of (177) and (127) that the moment of inertia of a body plays the same role, in rotational motion, as the mass of a body does in translational motion. Just as the mass of a body is a measure of the inertness of the body in the case of translational motion, so the *moment of inertia of a body about a given axis is a measure of the inertness of the body in its rotational motion about that axis.*

For one and the same rotational moment, the angular acceleration of a body is the smaller, the greater the moment of inertia of the body about the axis of rotation. An essential distinction of the moment of inertia of a body from its mass is, however, that the mass of the body is a constant quantity, whereas the moment of inertia of a body depends not only on the rotating mass itself but also on the distribution of that mass about the axis of rotation. For instance, take a long stick. It is much easier (Fig. 246) to bring the stick into rapid rotation by hand about its longitudinal axis, than about the axis perpendicular to its length (Fig. 247). This is due to the fact that in the first case the moment of inertia of the stick is appreciably less than in the second case. And so to impart the same angular acceleration in the first case requires a much smaller rotational moment.

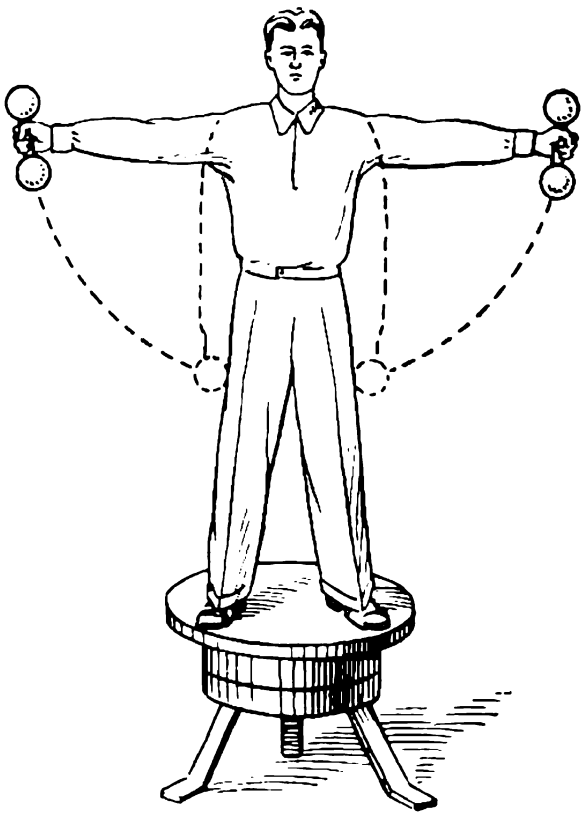


FIG. 248

This is well illustrated by means of stool revolving on a ball bearing (to reduce friction); it is called the Zhukovsky turntable. Referring to Fig. 248, a person standing on it when it is turning is able to change the angular velocity by changing the distribution of mass and thus by altering the moment of inertia of the system about the axis of rotation. The rotation of the system will speed up noticeably if the man brings his arms (with the weights) to his sides and, on the contrary, it will decrease if he brings the weights out to a distance from the axis of rotation of the system.

A wheel with a heavy rim and a light sleeve has a greater moment of inertia than a wheel of the same mass but with a reverse distribution of mass, that is, with a heavy sleeve and a light rim. This is because in the first case the greater part of the mass is at the greatest distance from the axis of rotation. And since the greater the moment of inertia of a body, the harder it is to change its motion, this is made use of in flywheels that serve to even out the running of a machine; they are large in diameter and the greater portion of the mass is on the rim.

Sec. 102. The moment of inertia of a body

To find the moment of inertia of a rigid body about an axis, say the z-axis (Fig. 249), partition the whole body into a large number n of very small particles, set up the sum of the products of the mass of each particle by the square of the distance of that particle from the given axis, and then compute the limit of this sum on the assumption that the number of particles, n, tends to infinity and the mass of each particle tends to zero:

$$J = \lim_{n\to\infty} (m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2) = \lim_{n\to\infty} \sum_{k=1}^{k=n} m_k r_k^2.$$

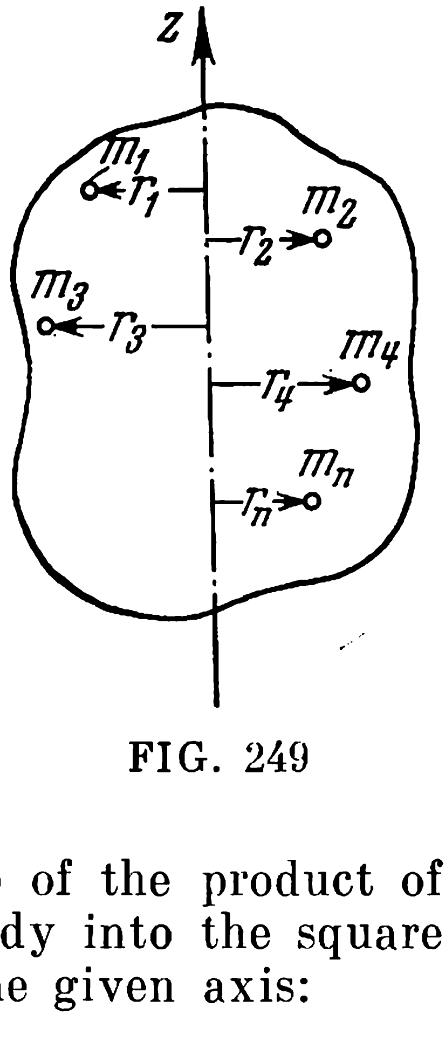


FIG. 249

Sometimes the moment of inertia J of a body about one of its axes is defined simply as the sum made up of the product of the mass m_k of each particle of the body into the square of the distance of that particle from the given axis:

$$J = \sum m_k r_k^2. \tag{178}$$

It is well to bear in mind that this is an approximate equation.

The moment of inertia obviously has the following dimensions:

$$[J] = [\text{mass}] \times [\text{length}]^2.$$

It is sometimes convenient to represent the moment of inertia J of a body about an axis in the form of a product of the mass M of the body into the square of the length of a certain line segment r_{in} called the *radius of inertia* of the body about the given axis:

$$J = Mr_{\text{in}}^2. \tag{179}$$

If the moment of inertia of a body about an axis is found (by computation or experiment), then the radius of inertia of the body about that axis is readily found from the preceding formula:

$$r_{\text{in}} = \sqrt{\frac{J}{M}}. \tag{180}$$

It is obvious that the radius of inertia of a body about a given axis may be understood as the length of a line segment equal to the radius from the given axis of a point at which the entire mass M of the body must be concentrated in order to obtain the moment of inertia of that point, which moment is equal to the moment of inertia of the body about the given axis.

Formulas for computing the moments of inertia of homogeneous bodies of diverse geometric shape may be found in engineering handbooks. Some of these formulas for certain homogeneous bodies of elementary geometric shapes are derived below, in Sec. *104. For inhomogeneous bodies or such that have complicated outlines, the moments of inertia are ordinarily found by experiment.

Problem 114. In order to find the moment of forces of friction in bearings, a flywheel of mass $M = 1000$ kg mounted on the shaft is given an angular velocity of $n = 300$ rpm. Left to itself, the flywheel comes to a halt after a lapse of $T = 100$ s. The radius of inertia of the flywheel is $r_{\text{in}} = 1.5$ m. Find the moment of friction, regarding it as constant.

Solution. According to the basic equation (177), for the rotational motion of a rigid body we have $M_{\text{rot}} = J\varepsilon$.

The rotational moment (moment of friction) of the flywheel is constant and so the rotation of the flywheel is uniformly variable. As we know, the angular acceleration of uniformly variable rotation is given by formula (104):

$$\varepsilon = \frac{\omega - \omega_0}{t}.$$

In this case, for $t = T = 100$ s, the angular velocity of the flywheel is $\omega = 0$, and the initial angular velocity (in radians per second) is

$$\omega_0 = \frac{\pi n}{30},$$

whence the angular acceleration of the flywheel is

$$\varepsilon = -\frac{\omega_0}{T} = -\frac{\pi n}{30T} = -\frac{3.14 \times 300}{30 \times 100} \approx -0.31 \text{rad/s}^2.$$

The minus sign here means that the rotation is decelerated.

We find the moment of inertia of the flywheel from formula (179):

$$J = Mr_{\text{in}}^2 = 1000 \times 1.5^2 = 2250 \text{ kg}\cdot\text{m}^2.$$

Hence the desired moment of friction is

$$M_{\text{rot}} = J\varepsilon = -2250 \times 0.31 = 697.5 \text{ N}\cdot\text{m}.$$

The sign of the moment of friction indicates that it is directed in opposition to the rotation of the flywheel.

Problem 115. A pulley (Fig. 250) is set in rotation by another (driving) pulley by means of a belt drive. The driving branch of the belt is stretched with a force of $T = 98$ N, the driven branch, with a force of $t = 49$ N. The mass of the pulley is $M = 200$ kg, the diameter is $D = 0.4$ m. Find the angular acceleration of the pulley and take into account the friction of the shaft in the bearings. The shaft has a diameter of $d = 0.06$ m and a coefficient of friction of $f = 0.1$. Disregard the weight of the shaft. Regard the pulley as a solid homogeneous cylinder.

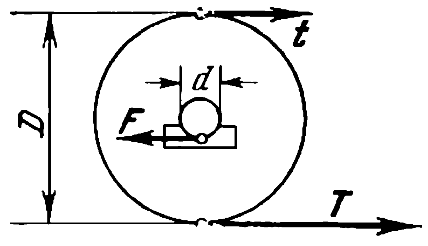


FIG. 250

Solution. The weight of the pulley is $G = Mg = 200 \times 9.81 = 1962$ N. The force of friction of the shaft in the bearings is

$$F = fG = 0.1 \times 1962 = 196.2 \text{ N}.$$

The rotational moment M_{rot} is equal to the algebraic sum of the moments of all forces (applied to the pulley) about the axis of its rotation:

$$M_{\text{rot}} = \sum m_z(\mathbf{P}_k) = T\frac{D}{2} - t\frac{D}{2} - F\frac{d}{2}$$

$$= (98 - 49)\frac{0.4}{2} - 196.2 \times \frac{0.06}{2} = 3.93 \text{ N}\cdot\text{m}.$$

The moment of inertia of the pulley (a solid homogeneous cylinder) is found from formula (184) given below in Sec. *104:

$$J_z = \frac{MR^2}{2} = \frac{MD^2}{8} = \frac{200 \times 0.4^2}{8} = 4 \text{ kg}\cdot\text{m}^2.$$

Knowing the rotational moment applied to the pulley and the moment of inertia of the pulley about its axis of rotation, it is easy to find its angular acceleration by using the basic equation (177) of dynamics for rotational motion of a body:

$$\varepsilon = \frac{M_{\text{rot}}}{J_z} = \frac{3.93}{4} \approx 0.98 \text{ rad/s}^2.$$

Sec. *103. A theorem on the moments of inertia of a body about parallel axes

The moment $J_{z'}$ of inertia of a body about some axis z' is equal to the moment J_z of inertia of that body about the z-axis passing through the centre of mass of the body and parallel to the given axis z' combined with the product of the mass M of the body into the square of the distance d between the axes.

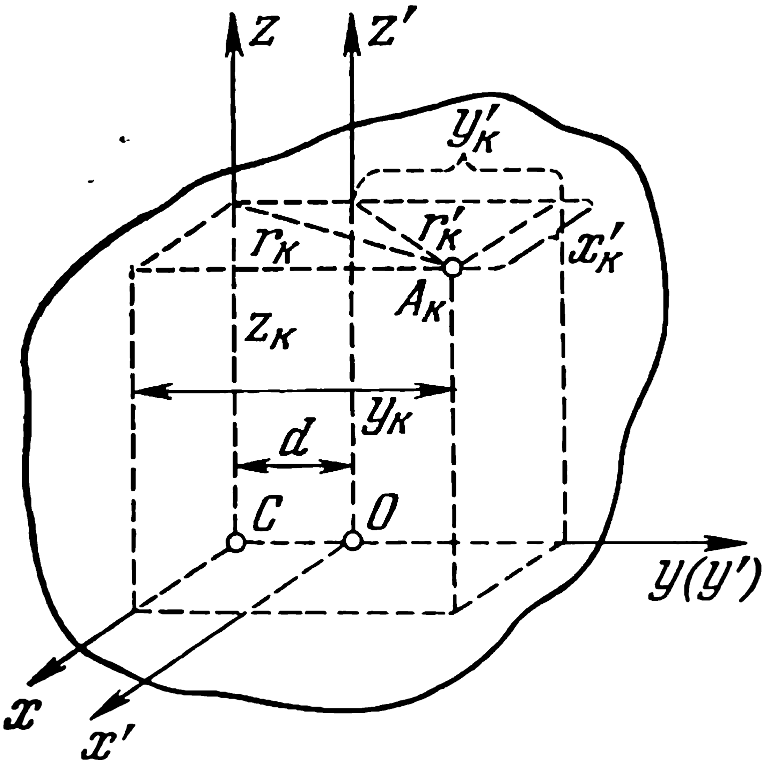


FIG. 251

Proof. For the centre of mass of the body, C, we take the origin of the system of coordinate axes $Cxyz$. We draw the axis Cy so that it intersects the axis z' in some point O, which we take for the origin of the new system of coordinate axes $Ox'y'z'$ (Fig. 251). Denote by d the distance between the axes Cz and Oz'. Now take an arbitrary particle A_k of the body of mass m_k. Denote by x_k, y_k, z_k the coordinates of this point in the system of coordinate axes $Cxyz$. Denote by x'_k, y'_k and z'_k the coordinates of the same point in the system of coordinate axes $Ox'y'z'$. Denote by r_k the distance of A_k from the z-axis, and by r'_k the distance of that particle from the z'-axis.

The moment of inertia of the body about the z'-axis is, by definition,

$$J_{z'} = \sum m_k r_k'^2.$$

But, as is evident from Fig. 251,

$$r_k'^2 = x_k'^2 + y_k'^2 = x_k^2 + (y_k - d)^2 = (x_k^2 + y_k^2) - 2dy_k + d^2$$
$$= r_k^2 - 2dy_k + d^2.$$

Substituting this value into the expression for $J_{z'}$, we get

$$J_{z'} = \sum m_k r_k'^2 = \sum m_k r_k^2 - 2d \sum m_k y_k + d^2 \sum m_k. \qquad \text{(I)}$$

Here, $\sum m_k r_k^2$ is the moment of inertia of the body about the central axis z.

From formula (171), which defines the ordinate y_C of the centre of mass, we find

$$\sum m_k y_k = My_C,$$

where $\sum m_k = M$ is the mass of the body. Since here $y_C = 0$, it follows that $\sum m_k y_k = 0$. Substituting these values into the right-hand side of (I), we get

$$J_{z'} = J_z + Md^2, \qquad (181)$$

and the proof of the theorem is complete.

If we know the moment of inertia of a body about some axis passing through its centre of mass, then, using (181), we can find the moment of inertia of that body about any other axis parallel to the given axis.

As follows from the same formula, the moment of inertia of the body about an axis passing through the centre of mass of the body is less than the moment of inertia of the given body about any other axis that is parallel to the given axis.

Sec. *104. The moments of inertia of certain homogeneous bodies of elementary geometric shape

1. **The moment of inertia of a thin straight homogeneous rod of constant cross-section.** A thin rod is understood to mean a cylindrical or prismatic body whose cross-sectional dimensions are small compared with its length.

(a) Referring to Fig. 252, let us compute the moment of inertia of a rod AB about the y-axis perpendicular to the length of the rod and passing through one end.

Partition the rod of length l into an infinite number of infinitesimal sections; denote by dm the mass of one such section of length dx at a distance x from the y-axis. If we denote the linear density of the rod, that is, the mass per unit length of the rod, by γ, then $dm = \gamma dx$. In accord with what was said above, set up the sum of the products of the mass of each section of the rod into the square of the distance of that section from the given axis; passing to the limit, we then obtain the definite integral

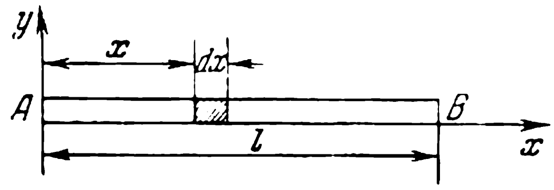


FIG. 252

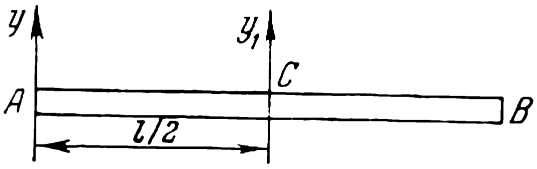


FIG. 253

$$J_y = \int_0^l x^2\,dm = \int_0^l \gamma x^2\,dx = \gamma \int_0^l x^2\,dx = \frac{\gamma l^2}{3}.$$

If we denote the mass of the whole rod by M, then this mass is $M = \gamma l$, whence the linear density of the rod is $\gamma = M/l$. Substituting this value of the density into the expression found for the moment of the inertia of the rod, we finally get

$$J_y = \frac{Ml^2}{3}. \tag{182}$$

The moment of inertia of a straight-line homogeneous thin rod about the axis perpendicular to the length of the rod and *passing through one end is equal to 1/3 the product of the mass of the rod by the square of its length.*

(b) Now let us find the moment of inertia of the rod about the axis y_1 (Fig. 253) that is perpendicular to its axis and passes through the centre of gravity of the rod (through the midpoint of the rod).

According to the theorem, proved above, on the moments of inertia about parallel axes, we have

$$J_y = J_{y_1} + Md^2.$$

Here, $J_y = Ml^2/3$ is the moment of inertia of the rod about the y-axis perpendicular to its length and passing through an end, J_{y_1} is the moment of inertia of the rod about the y_1-axis which is parallel to the y-axis and passes through the centre of gravity of the rod, M is the mass of the rod, and $d = l/2$ is the distance between the parallel axes y and y_1.

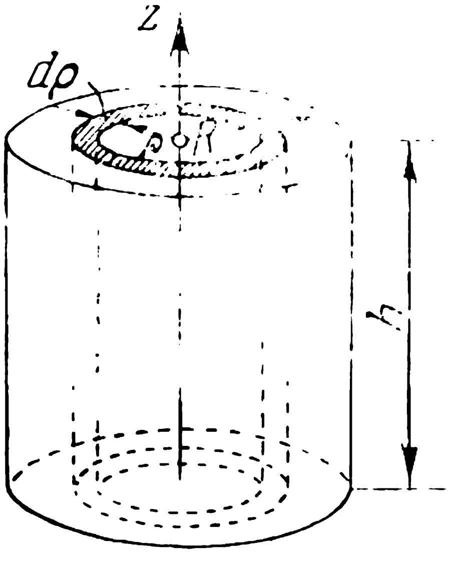


FIG. 254

Hence, the desired moment of inertia of the rod is

$$J_{y_1} = J_y - Md^2 = \frac{Ml^2}{3} - M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12}.$$

Thus we finally get

$$J_{y_1} = \frac{Ml^2}{12}. \tag{183}$$

The moment of inertia of a straight homogeneous rod about the axis perpendicular to the length of the rod and passing through its midpoint is equal to 1/12 the product of the mass of the rod by the square of the length.

2. **The moment of inertia of a circular solid homogeneous cylinder about its axis of rotation z.** Partition a cylinder of radius R and height h (Fig. 254) into an infinity of infinitely thin cylindrical layers; denote by dm the mass of one such cylindrical layer of radius ρ and thickness $d\rho$.

If we denote the density of the cylinder, that is, the mass per unit volume of the cylinder, by γ, then $dm = \gamma \times 2\pi\rho h d\rho$. Setting up the integral sum of the products of the mass of each layer of the cylinder into the square of its distance from the axis, we obtain

$$J_z = \int_0^R \rho^2\, dm = \int_0^R 2\pi\rho\gamma h\rho^2\, d\rho = 2\pi h\gamma \int_0^R \rho^3\, d\rho$$

$$= 2\pi h\gamma \frac{R^4}{4} = \frac{\pi R^4 h\gamma}{2}.$$

The mass of the whole cylinder is $M = \pi R^2 h\gamma$, whence the volume density of the cylinder is $\gamma = M/(\pi R^2 h)$. Substituting this value of the density into the expression just found for the moment of inertia of the cylinder about its axis of rotation, we finally get

$$J_z = \frac{MR^2}{2} = \frac{MD^2}{8}. \qquad (184)$$

The moment of inertia of a solid circular homogeneous cylinder about its axis of rotation is equal to half the product of the mass of the cylinder into the square of its radius.

3. **The moment of inertia of a homogeneous hollow circular cylinder about its axis of rotation.** Denote the outer radius of the hollow cylinder by R and the inner radius by R_0. Partitioning the hollow cylinder into an infinitely great number of infinitely thin cylindrical layers, we will clearly arrive at the same definite integral as in the preceding case, the only difference being the limits of integration.

Now, if the cylinder has a hollow of radius R_0, then the variable radius ρ of the cylindrical layers into which we partitioned the entire volume of the cylinder will not vary from zero to R but only from R_0 to R. Consequently,

$$J_z = \int_{R_0}^R \rho^2\, dm = 2\pi h\gamma \int_{R_0}^R \rho^3\, d\rho = 2\pi h\gamma \frac{R^4 - R_0^4}{4}$$

$$= \frac{\pi h\gamma (R^2 + R_0^2)(R^2 - R_0^2)}{2}.$$

Now the mass of a hollow cylinder is $M = (\pi R^2 - \pi R_0^2)\,h\gamma$, whence the volume density of this cylinder is

$$\gamma = \frac{M}{\pi (R^2 - R_0^2)\,h}.$$

Substituting this value of the density into the expression found for the moment of inertia of a hollow cylinder about its axis of rotation, we finally obtain

$$J_z = \frac{M (R^2 + R_0^2)}{2} = \frac{M (D^2 + D_0^2)}{8}. \qquad (185)$$

The moment of inertia of a homogeneous hollow circular cylinder about its axis of rotation is equal to half the product of the mass of the cylinder into the sum of the squares of its outer and inner radii.

In approximate calculations of the moments of inertia of hollow cylindrical bodies with a thin rim (for example, flywheels), the thickness of the rim is sometimes ignored and the whole mass of the body is considered to be uniformly distributed over its outer lateral surface. In that case, we put $R_0 = R$ in the preceding formula and get

$$J_z = MR^2 = \frac{MD^2}{4}. \qquad (186)$$

The moment of inertia of a thin homogeneous rim (a material circle) about its axis of rotation is equal to the product of the mass of the rim into the square of its radius.

Sec. 105. A theorem on the change in kinetic energy of a system

The kinetic energy of a system is the sum of the kinetic energies of all particles of the system.

Denoting the kinetic energy of a system by T, we have

$$T = \sum \frac{m_k v_k^2}{2}. \qquad (187)$$

Thus, to find the kinetic energy of a system at some instant we have to multiply the mass of each particle of the system by half the square of its velocity and then add the products.

Let us apply the theorem, given in Sec. 97, on the change in the kinetic energy of a particle to a system of such particles. For one particle we have equation (164):

$$\frac{mv^2}{2}-\frac{mv_0^2}{2}=A,$$

where $mv^2/2$ and $mv_0^2/2$ are the values of the kinetic energy of a particle at the end and beginning, respectively, of the path of the particle, A is the work of the force applied to the particle over that path.

Writing similar equations for all particles of the system and then combining them termwise, we will obviously get

$$\sum\frac{m_kv_k^2}{2}-\sum\frac{m_kv_{0k}^2}{2}=\sum A_k.$$

Denoting the kinetic energy of the system $\sum\frac{m_kv_{0k}^2}{2}$ in its initial position by T_0 and the kinetic energy of the system $\sum\frac{m_kv_k^2}{2}$ in its terminal position by T, we finally obtain

$$T-T_0=\sum A_k. \tag{188}$$

The change in kinetic energy experienced in displacing a system from one position to another is equal to the sum of the works of all forces (both external and internal) *acting on the system during that displacement.*

When calculating the sum of works of all forces acting on a system during its displacement, one should bear in mind the following circumstances* that in many cases greatly simplify the practical employment of the theorem on the change in kinetic energy.

1. *In any displacement of a solid body* (that is, a system the distances between the particles of which do not change), *the sum of the works of its internal forces is equal to zero.*

2. *The sum of the works of the reactions of time-invariable ideal constraints in any displacement of the system that is admitted by the constraints is equal to zero.*

---

* To demonstrate this would require reasoning that goes beyond the limits of our program.

This latter circumstance automatically eliminates from the equation of kinetic energy of a system the reactions of ideal constraints. *However, if the constraints are not ideal, that is, if the friction in them cannot be ignored, then the sum of the works of the forces applied to the system must include the work of the forces of friction.*

Sec. 106. The kinetic energy of a rigid body

When employing the theorem on the change in kinetic energy of a system, one very often has to calculate the kinetic energy of a moving rigid body. Let us find its expressions for the most important forms of motion of the body.

1. **The body is in translational motion.** If a rigid body is in translational motion, then the velocities of all its particles at any time are equal. Hence, in this case the kinetic energy of the body is

$$T=\sum \frac{m_k v_k^2}{2}=\frac{v^2}{2}\sum m_k.$$

Denoting $\sum m_k$, that is, the sum of the masses of all particles of the body, by M, we get

$$T=\frac{Mv^2}{2}. \qquad (189)$$

The kinetic energy T of a rigid body in translational motion is equal to half the product of the mass M of the body into the square of its velocity v.

And so, as we might have expected, the kinetic energy of a body in translational motion is calculated in exactly the same way as the kinetic energy of a particle.

Using formula (189), we can also calculate the kinetic energy of any system (for example, a belt thrown over a rotating pulley) moving in such a manner that the magnitudes of the velocities of all its particles are the same.

2. **The body rotates about a fixed axis.** The magnitude v_k of velocity of any kth particle of a rigid body rotating about a fixed axis is equal, as we know, to the product of the angular velocity ω of the body by the distance r_k

of the given particle from the axis of rotation of the body: $v_k = \omega r_k$.

Hence, the kinetic energy of a rotating body is

$$T=\sum \frac{m_k v_k^2}{2}=\sum \frac{m_k \omega^2 r_k^2}{2}=\frac{\omega^2}{2}\sum m_k r_k^2.$$

But $\sum m_k r_k^2$, that is, the sum made up of the products of the masses of the particles of the body into the squares of their distances from the axis of rotation, is nothing other than the moment of inertia J of the body about its axis of rotation.

Consequently, for the given motion of the body we have

$$T=\frac{J\omega^2}{2}. \qquad (190)$$

The kinetic energy T of a body rotating about a fixed axis is equal to half the product of the moment of inertia J of the body about that axis into the square of the angular velocity ω.

Comparing (190) with (189), it will be noted that their structure is similar. In (190), the role of the mass is played by the moment of inertia of the body, and the role of the linear velocity is played by the angular velocity of the body.

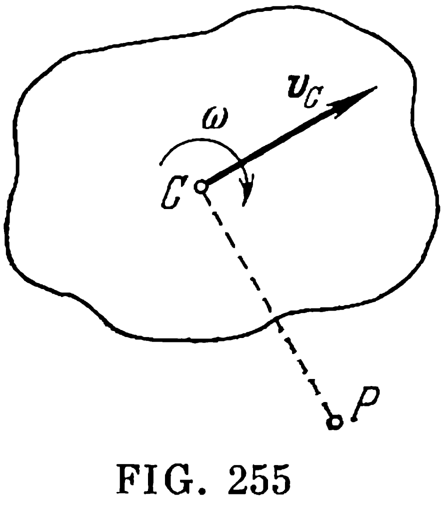


FIG. 255

***3. The body is in plane-parallel motion.** Suppose a body is in plane-parallel motion, that is, all the particles of the body are moving in planes parallel to a certain fixed plane. Imagine a cross-section of the body (Fig. 255) cut by a plane passing through the centre of gravity C of the body and parallel to the given fixed plane. Suppose we know the angular velocity ω of the figure and the velocity v_C of its centre of gravity.

Knowing this, we can readily find the position of the instantaneous centre P of the velocities of the figure. As we know from kinematics (Sec. 76), it lies on the perpendicular erected from some point of the figure to the direction of the velocity of the particle and at a distance equal to the ratio of the linear velocity of the given particle to the angular velocity* of the figure. Hence, the distance is

$$PC = \frac{v_C}{\omega}.$$

As was demonstrated in kinematics (Sec. 77), in the case of plane motion of a body, the velocities of its particles at every instant are distributed as if the body were in rotation, at that instant, about an instantaneous axis passing through the instantaneous centre of velocities of the figure (corresponding to that instant) and perpendicular to its plane.'

But the kinetic energy of the body $T = \sum \frac{m_k v_k^2}{2}$ depends solely on the mass of each particle and its velocity, and for that reason the kinetic energy of a body in plane motion may be computed from formula (190):

$$T = \frac{J_P \omega^2}{2}, \qquad \text{(I)}$$

where JP is the moment of inertia of the body about its instantaneous axis of rotation.

The use of formula (I) for finding the kinetic energy of a body in its plane motion is difficult due to the fact that it requires, for every instant of time, a determination of the position of the instantaneous axis of rotation of the body and a computation of the corresponding moment of inertia of the body. Let us therefore transform (I) by means of the theorem on the moments of inertia about parallel axes (Sec. *103). By this theorem,

$$J_P = J_C + M\,(PC)^2 = J_C + M\frac{v_C^2}{\omega^2}, \qquad \text{(II)}$$

where J_C is the moment of inertia of the body about the axis passing through the centre of mass of the body and parallel to the instantaneous axis, M is the mass of the body, and $PC = v_C/\omega$ is the distance between the parallel axes under consideration.

---

* As we know (Sec. 74), the angular velocity of a figure does not depend on the choice of pole.

Substituting expression (II), the expression for the moment of inertia J_P *of the body about the instantaneous axis, into (I), the formula for the kinetic energy, we find*

$$T=\frac{J_P\omega^2}{2}=\left(J_C+M\frac{v_C^2}{\omega^2}\right)\frac{\omega^2}{2}.$$

Removing brackets and cancelling, we finally get

$$T=\frac{Mv_C^2}{2}+\frac{J_C\omega^2}{2}. \tag{191}$$

The kinetic energy of a body in the case of plane motion is equal to the sum of the kinetic energies the body would have in the case of its translational motion with the velocity of the centre of mass of the body and in the case of its rotational motion about the axis passing through the centre of mass of the body and perpendicular to the fixed plane parallel to which the body is moving.

This statement can better be remembered if we recall that (see kinematics, Sec. 74) any plane motion may be decomposed into translational motion with velocity of the pole and rotational motion about the pole.

However, unlike kinematics, the choice of pole here is not arbitrary. When computing by (191) the kinetic energy of a body in plane motion, for the pole one must choose the centre of mass of the body. If a different point is chosen, then we get a formula for the kinetic energy that differs from (191).

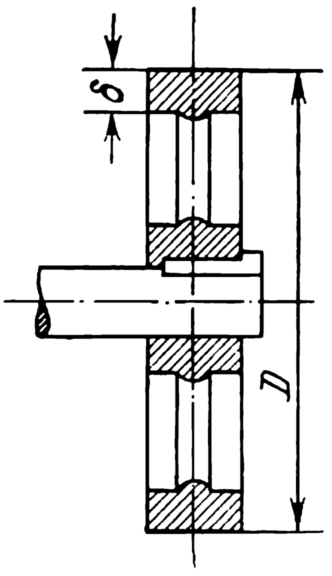


FIG. 256

Problem 116. A flywheel (Fig. 256) has an outer diameter $D = 4.6$ m; the weight of the rim is $G = 80$ kN, and the thickness of the rim $\delta = 30$ cm. What rotational moment must be applied to the flywheel for it to reach $n = 150$ rpm after $\varphi_{\text{rev}} = 200$ revolutions? Assume the moment of inertia of the spokes and sleeve of the flywheel to be 10% of the moment of inertia of the rim. Disregard the weight of the shaft and the friction in the bearings.

Solution. By the theorem on the change in the kinetic energy of a system [equation (188)],

$$T-T_0=\sum A.$$

We find the kinetic energy of the flywheel at the end of the period of work from formula (190): $T = J\omega^2/2$. Since the flywheel was stationary at the beginning, its kinetic energy at the initial time was

$$T_0 = \frac{J\omega_0^2}{2} = 0.$$

The change in the kinetic energy of the flywheel is due solely to the constant rotational moment applied to it, the work of that moment being equal, by formula (153), to

$$A = M_{\text{rot}}\varphi.$$

Thus, equation (188), the equation of the kinetic energy of the system, here assumes the form

$$\frac{J\omega^2}{2} = M_{\text{rot}}\varphi,$$

whence the rotational moment is

$$M_{\text{rot}} = \frac{J\omega^2}{2\varphi}.$$

The moment of inertia J_1 of the rim of the flywheel is found from formula (185) for a hollow circular cylinder:

$$J_1 = \frac{M(R^2 + R_0^2)}{2}.$$

The outer radius of the rim is

$$R = \frac{D}{2} = \frac{4.6}{2} = 2.3 \text{ m}.$$

The inner radius of the rim is

$$R_0 = \frac{D}{2} - \delta = 2.3 - 0.3 = 2 \text{ m}.$$

Hence the moment of inertia of the rim is

$$J_1 = \frac{G(R^2 + R_0^2)}{2g} = \frac{80\,000\,(2.3^2 + 2^2)}{9.81 \times 2} \approx 37\,880 \text{ kg}\cdot\text{m}^2.$$

The moment of inertia of the spokes and sleeve is

$$J_2 = 3790 \text{ kg}\cdot\text{m}^2.$$

The moment of inertia of the whole flywheel is

$$J = J_1 + J_2 = 37\,880 + 3790 = 41\,670 \text{ kg}\cdot\text{m}^2.$$

The angular velocity of the flywheel is

$$\omega = \frac{\pi n}{30} = \frac{\pi \times 150}{30} = 5\pi \text{ rad/s}.$$

The angle through which the flywheel turns after $\varphi_{\text{rev}} = 200$ revolutions is

$$\varphi = 2\pi\varphi_{\text{rev}} = 2\pi \times 200 = 400\pi \text{ radians}.$$

Substituting these data, we find the desired rotational moment:

$$M_{\text{rot}} = \frac{J\omega^2}{2\varphi} = \frac{41\,670 \times 25\pi^2}{2 \times 400\pi} \approx 4080 \text{ N}\cdot\text{m} = 4.08 \text{ kN}\cdot\text{m}.$$

Problem 117. A rope is thrown over a pulley (Fig. 257) of radius R and weight G_1. Suspended from the rope is a load weighing G_2. Initially the system is at rest. Find the angular velocity of the pulley at the instant when the load drops to a height equal to h. Regard the mass of the pulley to be uniformly distributed over its outer surface. Ignore friction.

Solution. Since the system was originally at rest, its initial kinetic energy T_0 is zero. The kinetic energy T of the system when the load is in motion will consist of the kinetic energy T_1 of the load itself and of the kinetic energy T_2 of the pulley that is set into rotation by the load. The kinetic energy of the load $T_1 = M_2v^2/2$, where $M_2 = G_2/g$. Since the velocity v of the load is connected with the angular velocity ω of the pulley by the relation $v = \omega R$, the kinetic energy of the load is

$$T_1 = \frac{M_2R^2\omega^2}{2} = \frac{G_2R^2\omega^2}{2g},$$

the kinetic energy of the pulley is

$$T_2 = \frac{J\omega^2}{2}.$$

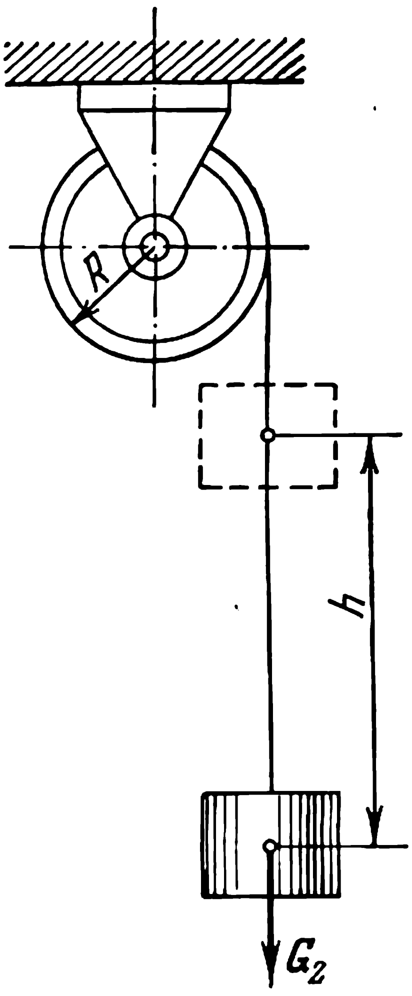


FIG. 257

The moment of inertia of the pulley is found from formula (186):

$$J = M_1R^2 = \frac{G_1}{g}R^2.$$

Hence,

$$T_2 = \frac{G_1R^2\omega^2}{2g}$$

and then the kinetic energy of the whole system is

$$T = T_1 + T_2 = \frac{G_2R^2\omega^2}{2g} + \frac{G_1R^2\omega^2}{2g} = \frac{\omega^2R^2}{2g}(G_1 + G_2).$$

Since the change in kinetic energy of the system occurs solely due to the work of gravity of the load dropping from height h, we have to put

$$\sum A = G_2 h$$

in the equation (188) of the kinetic energy of the system.

Thus, the equation for the kinetic energy of the system here takes on the form

$$\frac{\omega^2 R^2}{2g}(G_1+G_2)=G_2 h,$$

whence the desired angular velocity of the pulley is

$$\omega=\frac{1}{R}\sqrt{\frac{2ghG_2}{G_1+G_2}}.$$

Problem 118. A tank (Fig. 258) is set in motion by an engine that turns four wheels (two on each side). The wheels have projections that engage a caterpillar track. Find the speed of the tank after a lapse of 10 seconds from the start of motion if the average useful power of the engine is $N_{\text{av}} = 200$ kW, the mass of the tank is $M_1 = 10\,000$ kg, the mass of each caterpillar track is $M_2 = 700$ kg, and the mass of each wheel $M_3 = 200$ kg. Regard the wheels as solid homogeneous cylinders.

Solution. The system here consists of: (1) the body of the tank in translational motion, (2) two tracks, each of which is moving translationally together with the tank and at the same time is moving with the same velocity relative to the body of the tank, and (3) four wheels which are revolving on their axles and together with the axles are moving translationally, that is, are in plane-parallel motion.

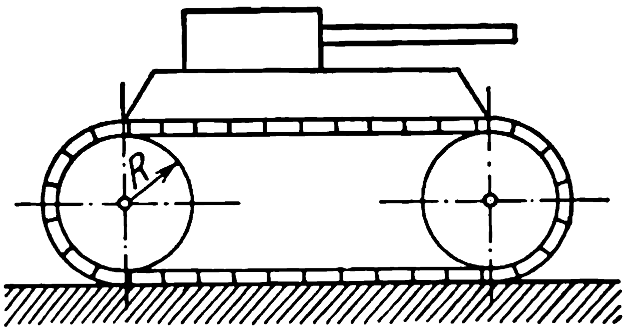


FIG. 258

The kinetic energy of the body of the tank is

$$T_1=\frac{M_1 v^2}{2}.$$

The kinetic energy of the tracks is

$$T_2=2\left(\frac{M_2 v^2}{2}+\frac{M_2 v^2}{2}\right)=2M_2 v^2.$$

The kinetic energy of the wheels is

$$T_3=4\left(\frac{M_3 v^2}{2}+\frac{J\omega^2}{2}\right).$$

The velocity of points on the axle of a wheel is equal to the translational velocity of the tank. And so the angular velocity of the wheels is

$$\omega=\frac{v}{R}$$

where R is the radius of a wheel.

The moment of inertia of a wheel (solid cylinder) is

$$J=\frac{M_3R^2}{2}.$$

Thus the kinetic energy of the wheels is

$$T_3=4\left(\frac{M_3v^2}{2}+\frac{J\omega^2}{2}\right)=4\left(\frac{M_3v^2}{2}+\frac{M_3R^2v^2}{2\times 2R^2}\right)=3M_3v^2.$$

The kinetic energy of the whole tank is

$$T=T_1+T_2+T_3=\frac{M_1v^2}{2}+2M_2v^2+3M_3v^2$$

$$=\frac{v^2}{2}(M_1+4M_2+6M_3).$$

Since at the start of motion of the tank its kinetic energy is $T_0 = 0$, the equation (188) of kinetic energy of the system takes on the form

$$T=\sum A.$$

Consequently, the useful work of the engine that sets the tank in motion is

$$\sum A=T=\frac{v^2}{2}(M_1+4M_2+6M_3).$$

The average useful power of the engine is

$$N_{\text{av}}=\frac{\sum A}{t}=\frac{v^2(M_1+4M_2+6M_3)}{2t},$$

whence the velocity of the tank 10 seconds after the start of motion is

$$v=\sqrt{\frac{2tN_{\text{av}}}{M_1+4M_2+6M_3}}=\sqrt{\frac{2\times 10\times 20\,000}{10\,000+4\times 700+6\times 200}}$$

$$\approx 16.9 \text{ m/s}=\frac{16.9\times 3600}{1000} \text{ km/h}\approx 60.8 \text{ km/h}.$$

INDEX

- absolutely rigid body 14
- acceleration 229
  - angular 274
  - centripetal 237, 278
  - normal 236, 239, 278
  - of a point 229ff, 241
  - rotary 278
  - tangential 235
  - true 231
- action and reaction, law of 23
- active forces 36
- addition
  - geometric 34
  - vector 34
- angle of repose 146
- angle of rotation 271
- angle of stability 202
- arc coordinate 210
- Archimedes 17
- Aristotle 17
- axiom of constraints 36
- axis 55
  - distance 212
  - of rotation 270
  - time 212

- balanced system of forces 31
- balancing force 33
- body, absolutely rigid 14

- carrier 339
- centre of figures and bodies of intricate shape 191
- centre of gravity
  - of a body 181, 182
    - coordinates of 183
  - of homogeneous bodies 187
  - of a line 185
  - of a symmetric body 186
  - of a volume 184
  - of volume of a prism 191
- centre of mass of a system 412
- Chaplygin, S. A. 20
- Chebyshev, P. L. 19
- circle
  - of curvature 238
  - osculating 238
- circular pitch 291
- classical mechanics 19
  - basic laws of 20
- conservation of momentum of a system, law of 409
- constraint(s) 35
  - axiom of 36
  - flexible 41
  - ideal 37
  - reactions of 35
- converging forces 43
  - plane system of 43
    - condition of equilibrium of 63
- coordinates
  - geographical 213
  - of a point 213
- Copernicus, N. 18
- cosine, direction 58
- Coulomb, Charles 135
- couple(s) 86, 87
  - arm of 88
  - composition of 94
  - equilibrium of 94
  - moment of 88
  - plane of action of 88
  - properties of 91
  - resultant of 95
  - rotation 336
- curvature
  - circle of 238
  - of a curve at a point 238
  - radius of 238

- d'Alembert, Jean L. R. 19, 357
- d'Alembert's principle 357
- delta 219
- derivative 222
- differential gears 339
- differentials 339
- differentiation, graphical 254
- direction cosine 58
- distance, initial 211
- driven members 287
- drive(s) 286
  - belt 287
  - friction 288
  - gear 289
  - plate-and-wheel variable-speed 289
- driving members 287
- dynamic force analysis 357
  - method of 358
- dynamic reactions of constraints 359
- dynamically balanced body 368
- dynamics 16, 344ff
  - basic equation 351
    - for rotational motion of a rigid body 420, 422
  - basic laws of 346ff
  - first problem of 345
  - of a point (general theorem on) 392ff
  - second problem of 345
- dynamometers 26

- efficiency 384
  - mechanical 386
- Einstein, Albert 21
- elastic force 381
- energy
  - kinetic 398, 399
    - change in (theorem) 400, 433
    - of a rigid body 435
  - potential 398
- equations of equilibrium 64, 112, 120, 170

equilibrium
  of a body with a fixed axis 173
  equations of 64, 112, 120, 170
  indifferent 200
  stability of 199ff
  stable 199
  of system of forces 165
  of system of linked bodies 123
  of three nonparallel forces (theorem) 64
  unstable 199
Euler, Leonhard 19

force(s) 24ff
  active 36
  arbitrary located in plane 98
  balanced system of 31
  balancing 33
  centre of parallel 178, 179
  centrifugal 361
  centripetal 361
  composition of two forces 43
  concentrated 114
  converging 43
  decomposing a force into components 46
  elastic 381
  external 31, 407
  of friction 40
  of inertia 357
  inertial 359
  internal 31, 407
  line of action of 26
  magnitude of 25
  moment of (about an axis) 161, 162
  normal 361
  parallelogram of 34
  point of application of 25
  polygon of 52
  reaction 36
  resolving a force into two parallel components 83
  rule of parallelepiped of 153
  spatial system of 152
  system of 30
  systems of two parallel 78ff
  triangle of 44
force polygon 52
friction 135ff
  angle of 143
  cone of 143, 144
  of the first kind 135
  forces of 43
  kinetic 137
  of motion 137
  pivoting 135
  rolling 135, 146, 148
  of the second kind 135
  sliding 135, 136
    coefficients of 139
  static 137
frequency of rotation 273
fulcrum 110

Galileo 18
  formula of 260

gear(s)
  bevel 292
  central 339
  differential 339
gear train, 295
gearing, multiple 293
graphical differentiation 254
graphical integration 255
gravity 181

heliocentric system of coordinates 347
horsepower 385
hp 385

idlers 296
impulse of a force 392
inertia 23
  force of 358
  law of 21, 23
  moment(s) of 425
    homogeneous bodies of elementary shape 430
    theorem on 428
  of a particle 360
inertial force 359
integration, graphical 255
intensity of a load 118
International Prototype Kilogram 354
International System of Units 354

joints 72
joule 371
Joule, J. P. 371

kgf 26
kilogram force 354
kilogram-force units 26
kilogram-force-metre per second 385
kilopascal 50
kinematics 16, 207ff
  of a particle (point) 209
kinetostatics 357

Lagrange, J. L. 19
law
  of equality of action and reaction 23, 351
  of independence of action of forces 352
  of inertia 21, 23, 346
Lenin, V. I. (definition of science) 16
lever 110
line of action of a force 26

magnitude of a vector 27
mass, definition of 349, 350
material point 14
mechanical efficiency 386

mechanical interaction 22
mechanical motion 13
mechanical work 371
mechanical system 345
mechanics 13, 17
  of bodies of variable mass 20
  classical 19
    basic laws of 20
  theoretical 13
Meshchersky, I. V. 20
method
  coordinate 213
  of cuts 127
  of isolating joints 72, 74
  of double projection 156
  of moment points 129
  natural (of specifying motion) 213
  stop 342
  Willis 342
module 291
moment
  of end-restraint reaction 116
  overturning 204
  of stability 203
momentum 392
  law of conservation 409
motion
  absolute 299, 310, 346
  compound 299, 310
  elementary types of (of a rigid body) 267ff
  equations of 212, 214
  harmonic oscillatory 356
  instantaneous translational 322
  law of 212
  mechanical 13
  migratory 299, 300
  nonuniform 210
  plane 310
  plane-parallel 310
  of a point
    rectilinear nonuniform 249
    special cases 247ff
    uniform 247
    uniformly variable 256
  relative 299
  resolving (into translational and rotational) 313
  translational 267
  rotational 270
    equation of 271
    special cases of 281
    transmission of 286
  uniform 210
  varying 210

n 273
N 26
newton 26, 354
Newton, Isaac 19, 354
newton-metre 89
normal
  to a curve 236
  to a surface 37
normal reaction 40

origin of a vector 27

Pa 50
parallelogram of forces 34
parallelogram rule 34, 304
particle 14, 21
pascal 50
Pascal, B. 50
path 210
perturbations 199
pitch circles 293
planet wheels 339
planetary gear trains 339
planimeter 376
Poinsot's theorem 98
point
  acceleration of 229ff
  material 14
  motion of (special cases) 247ff
  velocity of 218ff
pole 314
polygon of forces 51, 52
power 384
principal moment 99, 101, 165
principal vector 53, 99, 100, 165
principle
  of removability 36
  of rigidity 35, 94
projections
  of geometric sum of vectors 59
  of points 56
  of vectors 55ff

rack and pinion 292
rad/s 273
reaction force 36
recoil of a gun 411
reference frame 208
relativity, theory of 21
resultant 31
rolled stock 192
roller bearing 39
rotating rigid body 276
rotation(s)
  angle of 271
  axis of 270
  frequency of 273
  about parallel axes (composition of) 329
  uniform (equation of) 281
  uniformly variable 282
    equation of 283
    formula of angular velocity of 282
rotation couple 336
rotational moment 389
ratio (gear, speed, transmission, velocity) 290
rule of geometric addition 52

scalar quantities (scalars) 26
science, Lenin's definition of 16
SI 26, 354

stability
    angle of 202
    of a body on a plane 201
    coefficient of 204
    dynamic 202
    moment of 203
    static 203
static moment of area of a plane figure 183, 184
statically determinate problems 131, 133
statically indeterminate problems 131, 133
statics 16, 30ff
    axioms of 30
stiffness coefficient of a spring 382
strength of materials 19
sun wheel 339
system(s) 407
    of coordinates, right-hand 154
    equivalent (of forces) 31
    of particles 345, 407
        mechanical 345
    of two parallel forces 78ff
    of units 353ff
        engineering 354

terminus of a vector 27
theorem
    Poinsot's 98
    of three moments 113
    Varignon's 108
theoretical mechanics 14
thrust bearing 41
trajectory 210
triangle of force(s) 44
triangle rule 44
truss analysis 72
Tsiolkovsky, K. E. 20

unit(s)
    International System of 26, 354
    systems of 353ff
    technical (of mass) 354

Varignon, P. 109
Varignon's theorem 108
vector(s) 26
    absolute value of 27
    average velocity 219
    bound 28
    displacement 219
    free 28
    localized 28
    magnitude of 27
    modulus of 27
    principal 53
    projection of (on an axis) 56, 57
    quantity 26
    sliding 28
velocity (velocities)
    algebraic value of 225
    angular (of a vector) 337
    composition of 302
    distribution of (of points) 324
    instantaneous centre of 320, 321
    linear 276
    migratory 301
    numerical value of 221
    peripheral 277
    of a point 218ff
    projections of (theorem) 319
    resolving (of a point) 307

watt 385
Watt, James 385
watt-hour 385
weight 182
weightless rod 41
wheel(s)
    planet 339
    sun 339
Willis, R. 342
Willis method 342
Willis' formula 342
work
    of a constant force 369, 370
    of an elastic force 381
    elementary 374
    of force of gravity 378
    mechanical 371
    of a resultant (theorem) 376
    of a variable force 372, 374

Zhukovsky, N. E. 19
Zhukovsky turntable 424

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is:

USSR, 129820, Moscow I-110, GSP, Pervy Rizhsky Pereulok, 2, Mir Publishers

Printed in the Union of Soviet Socialist Republics